A significant correction to the 50-year-old theory for calculation of quasilinear diffusion coefficients used in heliophysics

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Kennel and Engelmann [1] published a quasilinear theory for wave-particle interactions that models the two-way coupling between waves generated by an unstable particle population, and scattering of the particles by those same waves. They derived diffusion coefficients in the 'resonant limit' where the wave growth goes to zero, and the diffusion coefficients were used to specify a Fokker-Planck equation that redistributes the particle phase space density in time as a function of the perpendicular and parallel components of velocity. The diffusion coefficients are linearly dependent on the distribution of electric field intensity, $\left|\vec{E}_k(\vec{k})\right|^2$, as a function of $\vec{k}$, the wave vector in 3D Cartesian coordinates, and thus the calculation obeys superposition as a function of the electric field wave intensity.

Lyons [2,3,4,5] adapted the approach in Kennel and Engelmann [1] to enable computation of the diffusion coefficients using a specification of the magnetic field wave intensity, $B^2(\omega)g(X)$, as a function of frequency, ω, and tangent of the wave normal angle, X, where $B^2(\omega)$ is the power spectral density and $g(X)$ is the wave normal angle distribution, here assumed to integrate to 1. Lyons [4,5] computes the distribution of the magnetic field wave intensity in wavevector space, $\left|\vec{B}_k(\vec{k})\right|^2$, using the formula

$$\left|\vec{B}_k(\vec{k})\right|^2 = \frac{B^2(\omega)g(X)}{N(\omega)}, \text{ where } N(\omega) = \frac{1}{(2\pi)^2}\int_0^\infty g(X)\left|k\cos^2\theta \left.\frac{\partial k}{\partial\omega}\right|_X\right| k_\perp dX \tag{1}$$

Glauert and Horne [6] noted that the integral in equation (1) defining the normalizing function, $N(\omega)$, diverges when an endpoint of the integral is equal to the tangent of the resonance cone angle, X_{rc}. They recommend setting the limit of the integral to $0.9999X_{rc}$ to avoid this problem.

Cunningham [7] noted that equation (1) does not obey superposition when $B^2(\omega)g(X)$ is generalized to $B^2(\omega)g_\omega(X)$ so that the wave normal angle distribution is allowed to be frequency-dependent, and identified two major errors in Lyons [4,5]. The first error is the fact that the azimuthal component of the wave vector is not integrated over properly and is not discussed further here. The second error is that the transformation from $\vec{k}$ to (ω, X) is done improperly. The correct relationship between two distributions of wave intensity, one in $\vec{k}$ coordinates and one in (ω, X) coordinates, is

$$\left|\vec{B}_k(\vec{k})\right|^2 = \frac{B^2(\omega)g_\omega(X)}{\frac{1}{(2\pi)^2}\left|k\cos^2\theta \left.\frac{\partial k}{\partial\omega}\right|_X\right| k_\perp} \tag{2}$$

where the denominator in equation (2) is the product of the two Jacobians needed to transform first from Cartesian wave vector coordinates to cylindrical and then from cylindrical wave vector coordinates to frequency and tangent of the wave normal angle. Cunningham [7] noted that equations (5) and (6) will produce nearly identical results when the Jacobian is nearly constant, and/or when the wave normal angle distribution is extremely narrow in X.

Equation (2) eliminates the issue noted by Glauert and Horne [6], since there is no integral needed to define the normalizing function in equation (2). The diffusion coefficients computed using equation (2) are shown to be well-defined even when the magnetic field intensity is nonzero at the resonance cone. This is because the intensity of the electric field near the resonance cone, where it becomes unbounded, is associated with extremely large wavenumbers that do not contribute to diffusion. Since the diffusion coefficients in Lyons [4,5] are linear in $\left|\vec{B}_k(\vec{k})\right|^2$, equation (2) will produce diffusion coefficients that obey superposition of wave intensities specified as function of (ω, X), as expected. Perhaps most importantly, equation (2) leads to diffusion coefficients that can be orders of magnitude different than those calculated using equation (1) when the wave normal angle distribution has nonzero values near the resonance cone. Large differences in diffusion coefficients are seen at lower energy when the ratio of the plasma frequency to the gyrofrequency is large, and at higher energies as this ratio decreases.

References

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