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Outline, Achievements and Recent Trends: Spectral Theory of Open Structures Dedicated to the 100 Anniversary of Victor P. Shestopalov

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Abstract

We briefly summarize the milestones describing comprehensive scientific acitivities and results achieved by Victor P. Shestopalov and his disciples in the fields specifically connected with applications of the nonselfadjoint operator spectral theory. We emphasize that his achievements paved the way to creating the modern level of the mathematical theory of wave propagation and diffraction.

1 Introduction

Scientific contribution of Victor P. Shestopalov to electromagnetics, modern mathematical theory of wave propagation and diffraction, and far beyond is enormous. In this short resume we can only cover a small part concentrating on his pioneering works devoted to the applications of the nonselfadjoint operator spectral theory and related items which in summary ended up with the creation of the spectral theory of open structures [1]. We note that the number of the works published by V. Shestopalov and his numerous disciples and collaborators in this field is measured by thousands; here we will cite only selected monographs and the very first works in the field as well as some of the most important published papers.

2 Milestones

There are clear milestones of the theory developed by V. Shestopalov that have been decisive in applications: (A) (nonselfadjoint) operators of the problems are considered in a mathematically correct manner (which has been elaborated as a result of many years of targeted research) as functions of the parameters, and (B) the boundary value problems (BVPs) for the Maxwell and Helmholtz arising in mathematical problems of wave propagation and diffraction in open structures are formulated correctly, particularly with the conditions at infinity that enable one to consider complex spectra and all types of solutions (among them the so-called surface, leaky, complex waves and oscillations and the like as well as bound states). Here the latter (B) is connected with the former (A) because the spectral parameter enters the conditions at infinity in a nonlinear manner.

3 Spectral Theory of Open Structures: Main Framework

3.1 Analytical Continuation of Solutions to Physical Problems

An essence of the key issue of the mathematical theory of open structures is in the following: the wave diffraction problems (which may be referred to as those with sources or excitation problems) considered initially at real frequencies are continued (extended) analytically to a certain multi-sheet Riemann surface H where the chosen spectral parameter is varied. In the end, the problem is reduced, for a wide family of open scattereres described in [1], to an operator equation in the form involving operator-valued functions (OVFs) $A(\lambda)$, $L+B(\lambda)$, or $I+D(\lambda)$ of the spectral parameter λ, which are of Fredholm or canonical Fredholm type on an appropriate pair of Banach spaces (here L is invertible and I is identical operator). This equivalent reduction performed in many cases using analytical regularization of operators [2] yields among all discreteness of the OFV spectra (i.e. discreteness of the sets of wave propagation constants or fundamental frequencies) which is extremely important both for constructing the methods of analytical and numerical solution and understanding the nature of wave processes. The (nonhomogeneous) diffraction problem is uniquely solvable everywhere in H except for a discrete set of eigenvalues of the corresponding homogeneous (spectral) problem (e.g. eigenfrequencies) forming a countable set of complex points with the only accumulation point at infinity; these points are finite-multiplicity poles of the analytical continuation of the operator of the initial diffraction problem and its Green function. Particularly, for cylndrical structures with compact two-dimensional cross-section, H may be chosen as the infinite-sheet Riemann surface of the analytical continuation of the fundamental solution $H_n^{(1)}(\lambda r)$ (Hankel functions of the first kind and order n) with respect to complex variable λ.

3.2 Existence and Distribution of Complex Spectra

Within this framework, one of the major tasks naturally arises of the spectral theory of open structures [1] in particular and mathematical theory of diffraction in general: the necessity of obtaining rigorous proofs of the existence of complex eigenvalues of the spectral problems (related by themselves to the analysis of oscillations and running waves) and description of their distribution in the complex domain. The sought eigenvalues are determined as the OVF characteristic numbers (CNs).

We have already mentioned that spectra of oscillations and running waves in open waveguides are studied in the mathematical theory of open structures using the reduction to spectral problems that describe singularities of the analytical continuation of the solution to the problems with sources to the domain of complex values of a parameter (frequency, longitudinal wavenumber). The resulting solution may not have direct physical meaning. However, this mathematically abstract approach has immediately demonstrated its efficiency in some canonical problems when it becomes possible to prove the existence and describe distribution of the spectra of eigenwaves or eigenfrequencies in the complex plane using closed-form solutions to forward scattering problems and analysis of operators as functions of several complex variables. Here one has to note the notion generalized dispersion equation (GDE) introduced first in [1] which represent generally functions or OVFs of several complex variables. GDEs appear in the well-known manner when eigenwaves or eigenoscillations are considered and the name dispersion equation (DE) is more commonly used. Here the word 'general' means that the number of unknowns in a GDE (dependent variables) and the number of independent variables may be different and greater than one or that GDE appears in an operator rather than ina functional form.

An abstract scheme and justification of the approach is given in Introduction to [1]. In short, the treatment of time-harmonic and nonstationary electromagnetic fields should be performed from the single viewpoint where the knowledge about the spectra is crucial. Namely, the solutions to the nonstationary inhomogeneous initial–bounday-value problem for Maxwell equations in its most general formulation can be asymptotically represented as series in terms of eigenfunctions of the corresponding Maxwell operator, and the time evolution (and thus the correct classification) of a free oscillation can be described by eigenfunctions and eigenvalues of the Maxwell operator, where eigenvalues may be arbitrary complex numbers, if one can effectively determine its point spectrum.

In fact, the spectra are the singularities (poles) of the resolvent of the forward problems and can be investigated in terms of the analysis of scattering matrices determined explicitly. This is performed using summation equations and operators represented by infinite matrices [4, 5, 10]. In several cases, it becomes possible to prove the existence of infinite complex sequences of these poles and describe their distribution and asymptotics in the complex plane following classical results of the spectral theory of linear differential operators. In particular, the occurrence of spectral singularities can be explained using the proposed approach and they can be classified as real subsets of the sets of complex eigenfrequencies. Other important findings are the continuous dependence of singularities on the problem parameters, the possibility of their efficient calculation using numerical solution to closed-form transcendental equations in the complex domain, and extension of the obtained results and methods to layered dielectric structures with arbitrary number of layers and more complicated scatterers. The findings have led to the determination of scattering frequencies of 2D and 3D metal-dielectric structures (slotted cavities and guides with inclusions).

3.3 Recent Developments

Rigorous proofs of the existence of complex resonance singularities of the solution to the problem of the plane wave scattering by circularly symmetric open structures (like a homogeneous dielectric cylinder of circular cross section partially covered by a perfectly conducting screen, the so-called cylindrical slot line) proceed from a mathematically correct approach of the spectral theory of open structures [1, 10] involving generalized conditions at infinity [3] that enable one to handle complex resonance frequencies. The main idea of the analysis is as follows: if one of the series solution coefficients considered as a function of all three problem parameters is singular then the whole solution is singular. The present approach leading to the rigorous proofs employs the technique set forth for the determination of complex waves in metal-dielectric waveguides [13, 14].

4 Spectral Theory: Operator-Valued Functions Nonlinear With Respect to the Spectral Parameter

The main achievements of V. Shestopalov and his numerous disciples are in the fields of the application of the methods of the spectral theory of OVFs, summation equations, singular integral equations (SIEs), and in the end pseudo-differential operators (PDOs) for the analysis and development of the solution methods for BVPs for the Maxwell and Helmholtz equations in three-dimensional (3D) domains with obstacles of complicated structure including unbounded spatial domains with noncompact multi-connected boundaries (guides and layered structures with large inclusions and almost-periodic boundary perturbations stretching to infinity).

The developed technique employing OVFs involves determination of CNs of the Fredholm operator-valued functions of the problems constructed in many cases using Green's potentials. Separating principal parts in the form of meromorphic operator pencils the operator generalization of Rouche theorem is applied to verify the occurrence of CNs in close proximities of the singularities of the pole pencils.

5 Critical Phenomena. Interaction of Oscillations

A huge amount of results obtained beginning from the late 1980s by V. Shestopalov and his collaborators tackles [15] the development of the methods of spectral theory of OFVs and perturbation theory for investigation of various critical phenomena based on the analysis of the eigenvalue degeneration and critical points (CPs) in the eigenvalue dependence on parameters. The developed methods have been applied to the analysis of critical regimes occurring in large-scale computations and description of complicated resonance and interaction phenomena in electromagnetics and acoustics, including existence and distribution on the complex plane of scattering frequencies; interaction of oscillations and waves in open resonators and waveguides; and wave propagation in nonlinear media. Description and simulation have been performed of the interaction phenomena in open resonators, waveguides, and transmission lines based on the analysis of CPs in the operator dependence on parameters.

6 Historical Notes

Correct mathematical statements and justification of the series solution for cylindrical structures with circular symmetry using the spectral (Fourier) method was first accompished in [4, 5]. These papers as well as several others cited therein co-authored by V. Shestopalov and published in the 1980s may be considered as pioneering spectral theory of open structures where the technique applying nonlinear operator spectral theory methods and OVFs was developed. This approach was later enahanced and summarized in [1, 11]. An outstanding collection of the detailed methods of analysis of the summation equations in electromagnetics can be found in [10]. The studies [1, 11, 12] published as monographs internationally were performed in parallell with other scholars, like [6], and yielded a flow of research of the topics related to the spectral theory of open structures reflected in the books [7, 8, 9] to name a few and many other works. The amount of subsequent publications of different authors developing spectral theory of open structures since the 1980s my be estimated today to be as high as several thousand.

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