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On a Method of Investigating Nonlinear Dispersion Relations of Electromagnetics in the Complex Domain

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Abstract

We explore the notions of generalized dispersion relations (DRs) and equations and develop on this basis an approach to investigate nonlinear DRs arising in many typical electromagnetic settings. The technique employs analysis of implicit complex-valued functions of several complex variables and implicit compex mappings and generalizes the known methods of the DR analysis.

1 Introduction

Generalized dispersion relations (GDRs) in electromagnetics introduced first in [1] represent generally functions of several complex variables. GDRs appear in the well-known manner when eigenwaves or eigenoscillations are considered and the names dispersion equation (DE) or generalized DE (GDE) are more commonly used. Here the word 'general' means that the number of unknowns in a GDE (dependent variables) and the number of independent variables may be different and greater than one.

The less popular case is when the expansion coefficients $A_n(\mathbf{w})$ of the scattered field components $\phi(r,\theta)$ in the representations of the type

$$\phi(r,\theta) = \sum_{n=-\infty}^{\infty} A_n(\mathbf{w}) H_n^{(1)}(k_0 r) e^{in\theta}, \tag{1}$$

are considered [2, 3] as functions of the problem parameters; here (r,θ) are polar coordinates, $H_n^{(1)}$ are Hankel functions of the first kind and order n, and $\mathbf{w} = \{w_j\}_{j=1}^{L}$ is the parameter vector that characterizes the structure of the scatterer cross-section. Then, the relations for zeros or singularities of $A_n(\mathbf{w})$ may be treated as GDRs or GDEs.

2 Generalized Cylindrical Polynomials and Reference Points

For a wide family of circular-symmetric piecewise homogeneous metal–dielectric scatterers (the benchmark among them are a dielectric rod and one- and multi-layer Goubau lines) DRs and the expansion coefficients in (1) can be obtained in the closed form. What is more, it turns out that they can be expressed using the so-called generalized cylindrical polynomials (GCPs) [4]. A GCP $G_N(\mathbf{w})$ has the form

$$G_N(\mathbf{w}) = \sum_{n=1}^{N} \mu_n(\mathbf{w}) \mathscr{I}_n(\mathbf{w}), \ \mathscr{I}_n(\mathbf{w}) = \Pi_{m=1}^{N_n} I_m(\mathbf{w}), \tag{2}$$

where μ_n are weights (constants or bounded continuous functions of its argument), $\Pi_{m=1}^{N_n}$ denote the products of N_n factors, and $I_m(\mathbf{w})$ are cylindrical functions of order m[1]. If μ_n are real and I_m are real-valued, e.g. Bessel or Neumann, functions, then each term in (2) has infinitely many real zeros and GCP $G_N(\mathbf{w})$ itself has [4] real zeros alternating with the zeros of its individual terms. The so-called reference points (RPs) $\mathbf{w}^*$ at which $G_N(\mathbf{w}^*) = 0$ may be thus determined explicitly (as a collection of zeros $z_{m'}^{N_{n'}}$ of the individual terms $\mathscr{I}_n$ at which at least one its factor $I_{m'} = 0$, $1 \le m' \le N_{n'}$) if e.g. dim $\mathbf{w} = L = N$ and the system $\{\mathscr{I}_n(\mathbf{w}) = 0, 1 \le n \le L$ has a solution. However, if some I_m are complex-valued, e.g. Hankel, functions and have complex zeros, the RPs may be complex

3 Implicit Complex-Valued Functions and Mappings and Generalized Dispersion Relations

In expansions (1) coefficients $A_n(\mathbf{w})$ may have zeros and poles which describe resonances or partial invisibility achieved due to the suppression of several scattered-field harmonics. The sought quantities that enter GDE, GDR or expressions for $A_n(\mathbf{w})$, like natural frequencies or the wave propagation constants (often referred to as spectral), are generally determined as implicit complex-valued functions of the remaining parameters describing the scatterer (often called nonspectral).

The key issue and main task of the theory is rigorous proof of the existence of these implicit complex-valued functions (or, if expressed in a simpler manner, existence of the GDE or GDR zeros, or poles and zeros of the scattered field expansion coefficients) and determination of the location of the corresponding complex hypersurfaces in the multidimensional complex space. In fact, numerical tools and

[1]Generally, GCPs are introduced recursively, so that certain I_m in (2) may be GCPs

solvers cannot be used for this purpose and must be complemented by the exact information of this sort obtained theoretically. The simplest possible situation here is when one looks for the dependence of one real spectral quantity versus one real nonspectral one which ends up with the simplest hypersurface: a dispersion curve (DC) on the phase plane of two real variables.

Recent findings pave the way for obtaining solutions to these problems in terms of the analysis of GCPs [4]. The GCPs-based method enables one to explicitly determine the RPs and then to reduce the determination of implicit complex-valued functions to the solution of the initial-value (Cauchy) problems for ordinary differential equations or their systems with the initial conditions that employ information about the found reference points. This is a clear path to the desired existence and simultaneously provides with an analytical technique to calculate the sought quantities. A more general approach proposed recently is to consider implicit complex-valued functions associated with investigation of GDE, GDR, and expansion coefficients as *complex mappings* that provide an image (a domain rather than single curves, DCs) of a set of the complex parameter plane on the complex plane of the sought quantity.

4 Example: Circular Dielectric Rod

For a circular dielectric rod of radius a the parameter sets [2] at which $g_n(\kappa,\zeta)=0$, $n=0,1,2,\ldots$, where

$$g_n(\kappa,\zeta)=J_n(\zeta\kappa)H^{(1)}_{n+1}(\kappa)-\zeta H^{(1)}_n(\kappa)J_{n+1}(\zeta\kappa) \qquad (3)$$

is a GCP of order 2, constitute resonace modes, so that for every fixed ζ (the refraction index) the corresponding values of $\kappa=k_0a$, where k_0 is the wavenumber of vacuum, are eigenfrequencies (poles of the scattered field expansion coefficients); here J_n are the cylindrical (Bessel) functions of the index (order) n and $\mathbf{w}=\{w_j\}_{j=1}^{L=2}=(\kappa,\zeta)$. Clearly, the RPs are $\mathbf{w}^*=(\kappa^*,\zeta^*): J_n(\zeta^*\kappa^*)=0, H^{(1)}_n(\kappa^*)=0$. The GDEs $g_n(\kappa,\zeta)=0$, $n=0,1,2,\ldots$, define $\kappa=\kappa(\zeta)$ as implicit complex-valued functions that may be considered as implicit complex mappings $\kappa(\zeta): \mathrm{C}_\zeta\to\mathrm{C}_\kappa$ and determined as solutions to the Cauchy problems for ordinary differential equations in the complex domain $\kappa'=\Phi(\kappa,\zeta)$ with the initial conditions $\kappa(\zeta^*)=\kappa^*$. Remarkably, $\Phi(\kappa,\zeta)$ is a ratio of two second-order GCPs (because one can call that partial derivatives of GCPs are GCPs as well), which makes it possible to show that the existence of the sought implicit functions $\kappa=\kappa(\zeta)$ (i.e. the existence of the corresponding resonance frequencies) follow form the fact (verified explicitly [2]) that $\frac{\partial g_n}{\partial\kappa}(\mathbf{w}^*)\neq 0$.

Figure 1 exemplifies action of such a mapping demonstrating a domain of C_κ which is a result of approximate transformation of the disk $\{\zeta: |\zeta-\zeta^*|<0.06\}\subset\mathrm{C}_\zeta$ by $\kappa=\kappa(\zeta)$. One can see that the diameter of the transformed disk in the κ-plane is by more than an order of magnitude greater than that of the parameter domain in the ζ-plane,

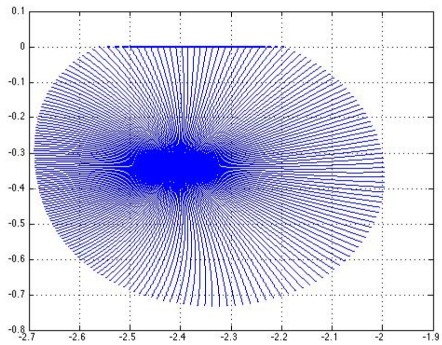

Figure 1. Approximate image under the action of $\kappa=\kappa(\zeta)$ of a vicinity of the RP component $\zeta: |\zeta-\zeta^*|<0.06$ on the complex plane C_κ, where $\zeta^*=\nu_1^0/\kappa^*$, ν_1^0 is the minimal positive zero of the zeroth-order Bessel function J_0, $J_0(\nu_1^0)=0$, and κ^* is a complex zero of the Hankel function $H_0^{(1)}$ with the minimal absolute value situated in the third quadrant of C_κ.

so that the initial domain affected by $\kappa(\zeta)$ actually 'blows up'. Each visible curve in the figure is a separate DC obtained as a result of the parametrization of variable ζ and then correspondingly of $\kappa(\zeta)$ by one real parameter.

5 Conclusion

Developing the GDR technique aimed at various electromagnetic applications and beyond we have proposed to enhance this method by considering implicit complex-valued functions of several complex variables and implicit compex mappings, often replacing standard DCs, that carry complete information about the resonance and scattering phenomena. We have demonstrated that implementation of the approach may be fully performed by clear numerical procedures of solving initial-value problems with the intitial conditions determined explicitly.

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