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Fields Excitation in Open Resonators based on Shestopalovs - Poedinchuk Spectral Theory

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Abstract

A new approach to solving the problems of excitation of open resonators (OR) by internal sources is proposed and substantiated. On the basis of the Shestopalovs-Poedinchuk Spectral Theory of OR, an expression for the complete Green's function of a two-dimensional OR formed by ideally conductive cylindrical mirrors is obtained. Equations describing the complex amplitudes of resonant modes and, at the same time, allowing to estimate the non-resonant radiation of the source from the OR are derived. The general structure and nature of the dependence of the non-resonant part of the Green's function in the vicinity of a separate resonance on the source frequency have been studied. Equations for slow-varying amplitudes of resonant modes are obtained, suitable for describing a single-mode and multimode self-oscillations in the generators with OR.

1 Introduction

The main difficulty in constructing the theory of excitation of open resonators (OR) by internal sources is that, unlike microwave cavities, there are no normal modes for them that form a complete system of eigenfunctions of the discrete spectrum, and, consequently, representations of the excited field in the form of the superposition of these modes with coefficients calculated through source currents. From a physical point of view, this is due to the fact that in addition to the diffraction energy losses of OR resonant modes, in the ORs there is always take place non-resonant radiation of the source, which is not associated with the energy loses of these modes [1, 2]. When analyzing forced oscillations in ORs using the well-known excitation theory [3], it is necessary to use solutions of the corresponding spectral problems obtained in the approximation of the parabolic equation. These solutions are valid in the short-wave approximation and only approximately describe the fields outside the resonant volume Vp bounded by the OR mirrors and caustic surfaces, especially when the characteristic dimensions of the OR are comparable to the wavelength of the excited oscillations.. Therefore, the OR excitation theory suggested in [3] does not actually allow us to study the radiation fields from the resonant volume and calculate the loss of source energy on their excitation.

In the present paper, a new approach is developed in the theory of excitation of two-dimensional ORs by internal sources, the parameters of which may nonlinearly depend on the excited fields. The proposed theory is based on effective algorithms followed from rigorous method of the Riemann–Hilbert problem [4] and used for constructing the Green's function for the considered class of ORs [4-6] as well as for solving the related spectral problem [5-7]. The solutions to the excitation problem found describe the field both inside and outside the resonant volume with an arbitrary relationship between the wavelength and the dimensions of the OR. In addition, non-stationary equations for slowly changing fields are obtained.

The construction of the theory of excitation of OP by given currents, which makes it possible to take into account the loss of the source for non-resonant radiation, is based on the use of well-developed strict [4-7]. In this case, it is possible to mathematically correctly pose and solve the spectral problem [5-7] and, with the help of the Green function, to build a solution to the excitation problem, as well as to establish their relationship [8]. In this case, it is possible to mathematically correctly pose and solve the spectral problem [5-7] and, with the help of the Green's function, to build a solution to the excitation problem, as well as to establish their relationship [8].

2. Problem Statement

Consider the OR formed by two circular cylindrical screens S_1 and S_2 (Figure. 1) with arbitrary both the radii of curvature and the distance between them. Electromagnetic fields are excited by an internal source with a single non-zero component of the current density, $j_z(x, y)$, oscillating with frequency ω. In this case, E-modes will be excited in the OR, which have the longitudinal component of the electric field, E_z. This allows us to confine ourselves to considering the Helmholtz scalar equation derived from Maxwell's equations:

$$\Delta E(x, y) + k^2 E(x, y) = -j_\omega(x, y) \qquad (1)$$

where $j_\omega = i\frac{4\pi k}{c} j_z$ is Fourier image of the electric current density of the source; $k = \omega/c$, c – velocity of light.

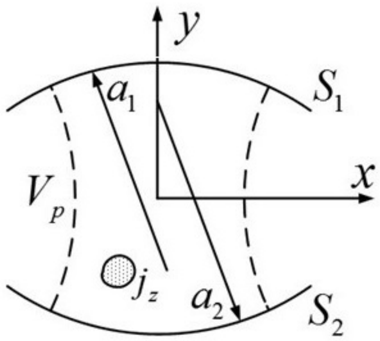

Figure 1. Open Resonator formed by two cylindrical screens S_1 и S_2 having radii a_1 and a_2 and containing source of electric current.

The desired function $E(x,y)$ must be equal zero on mirrors S_1 and S_2, satisfy the finite field energy condition in a limited volume (Meixner's condition) and the outgoing radiation condition [5-7]:

$$E(x,y) \sim \sum_n a_n H_n^{(1)}(kr)\, exp(in\varphi) \qquad (2)$$

for $|kr| \gg 1$, where $r^2 = x^2 + y^2; cos\,\phi = \frac{x}{r}; sin\,\phi = \frac{y}{r}$; $H_n^{(1)}(\dots)$ is Hankel function of the first kind of nth order.

The condition (2) is called the Reinhardt condition or the outgoing radiation condition because it means that the eigenstate field at infinity must have the form of a superposition of diverging cylindrical waves carrying away the energy of the field.
The desired solution of quasilinear equation (1) is conveniently represented using the Green's function $G(x,y,x_0,y_0)$ of the Helmholtz equation:

$$E(x,y) = \iint G(x,y,x_0,y_0) j_\omega(x_0,y_0) dx_0 dy_0 \qquad (3)$$

The formal transition from (1) to (3) indicates only the path of analysis of the formulated task. To advance in its solution, one has to consider the corresponding spectral problem, build a Green's function and obtain equations for the amplitudes of the field.

3. Spectrum of OR Resonant frequencies, Green's function, Resonant fields and non-resonant radiation

The Spectral problem, the problem on the spectrum of OR eigenstate modes, consists in determining the values of the spectral parameter k ($k = \omega_0/c$, ω_0 is the resonant frequency of the OR) at which a non-trivial solution of the homogeneous Helmholtz equation satisfying the condition (2) do exist. Using the Riemann–Hilbert problem method, it can be shown [5-7] that the formulated spectral problem is equivalent to the problem of characteristic numbers of the canonical Fredholm operator-function, which has the form [4-7]:

$$\begin{Vmatrix} I - A^{11}(k) & -A^{12}(k) \\ -A^{21}(k) & I - A^{22}(k) \end{Vmatrix} : l_2 \times l_2 \to l_2 \times l_2 \qquad (4)$$

where I is an identical operator. Via $A^{ij}(k)$ the matrix operator-functions analytically dependent on k and operating in Hilbert space of infinite number sequences are indicated: $l_2 = \{(z_n)_{n=-\infty}^{+\infty} : \sum_n |z_n|^2 < \infty\}$; $A^{ij}(k): l_2 \to l_2$.
Explicit view of $A^{ij}(k)$ is given in [5-7]. From the spectral theory of core operator-functions in Hilbert spaces, it follows that the set of characteristic numbers of the operator-function (4), and hence the spectrum of $\sigma(\Delta)$ resonant frequencies of the OR, is *discrete*, *finite* and lies in the *region with* $Imk < 0$. The only accumulation point for a set of $\sigma(\Delta)$ is the infinitely distant point: $|k| \to \infty$. To each resonant frequency $k_s \in \sigma(\Delta)$ corresponds its eigenstate function $g_s \in \sigma(\Delta)$, which describes the field not only within the resonant volume, but throughout space. Eigenfunctions in this case are built numerically.

Let's move on to the construction of the Green's function [8]. Let's place the investigated OR in a medium with absorption ($Im\,\varepsilon > 0, Im\,\mu > 0$) and introduce the designation $k = \frac{\omega}{c}\sqrt{\varepsilon\mu}$, ε and μ the dielectric and magnetic constants of the medium, respectively. Then the Green's function is a solution to the following equation:

$$\Delta G + k^2 G = -\delta(x - x_0)\delta(y - y_0) \qquad (5)$$

satisfies the Meixner condition and the infinity condition: $G(x,y,x_0,y_0) \to 0 \;\; at \;\; r \to \infty$. The Green function is searched as $G = G_0 + G_1$, where $G_0 = -\frac{i}{4} H_0^{(1)}(k|r - r_0|)$ and, G_1 satisfies the homogeneous Helmholtz equation and the heterogeneous boundary condition on the mirrors S_1 and S_2: $G_1 = -G_0$. Construction of the Green's function for equation (1) with the help of the Riemann-Hilbert method [4] is reduced to solving a system of linear operator equations of the second kind, having the form [5-8]:

$$\begin{Vmatrix} z^1 \\ z^2 \end{Vmatrix} = \begin{Vmatrix} A^{11}(k) & A^{12}(k) \\ A^{21}(k) & A^{22}(k) \end{Vmatrix} \times \begin{Vmatrix} z^1 \\ z^2 \end{Vmatrix} + \begin{Vmatrix} b^1 \\ b^2 \end{Vmatrix} \qquad (5)$$

where $z^j = (z_n^j)_{n=-\infty}^{+\infty}$ are unknown coefficients of decomposition of the function G_1 into a Fourier series over the angular coordinate φ_j of the local coordinate system associated with the mirror, S_j; $b^j = (b_n^j(x_0,y_0))_{n=-\infty}^{+\infty}$ - are similar coefficients of decomposition of the point source function, depending on its coordinates x_0, y_0. With $Im\,k > 0$ for any $b^j \in l_2$. the solution of equations set (5) exists and unique [5-8]. The solution we need for the real eigenstate frequencies ($Im\,k = 0$) is understood as the limit: $G_1(k))$ at $Im\,k \to 0 + o$.

For convenient representation of the Green's function, let's use the fact that the matrix operator in (5) is a core type one: $\sum_n A_n^{ij}(k) < \infty$ [5-8]. Core matrix operators allow for the introduction of infinite determinants analogous to the determinants of finite-dimensional matrix operators. Consequently, the solution of equations of the form (5) may be presented according to Kramer's rule:

$$z_n^i(x_0, y_0) = D_n(k, b^1, b^2)/D(k),$$

where $D_n(k, b^1, b^2)$ is the determinant obtained from the determinant D(k) of the operator (5) when replacing the elements of the $n-th$ column with the corresponding column in right side, b^1, b^2. Then:

$$G(k,x,y,x_0,y_0) = G_0(k,x,y,x_0,y_0) + \sum_{j=1}^{2}\sum_{n} \frac{D_n(k,b^1,b^2)}{D(k)} G_n^j(k,x,y) \quad (6)$$

where

$$G_n^j(k,x,y) = e^{in\phi_j} \begin{cases} J_n(ka_j)H_n^{(1)}(kr_j), r_j > a_j \\ H_n^{(1)}(ka_j)J_n(kr_j), r_j \le a_j \end{cases}$$

From (3) and (6), it is seen that the constructed Green's function gives the required behavior of the EM field at infinity that satisfies the condition (2). The constructed Green's function allows calculation of the full field excited by the internal source. The presence of even relatively weak resonances in the system makes it possible to isolate its resonant part in the full field. The intensity of this resonant part of the field is higher the closer the frequency of the source to the related resonant frequency and the greater the *Q*-factor of the resonance. From the expression (6) it can be seen that with the analytical continuation of the Green's function in the lower half-plane, this function has features of the pole type at the points k_s, coinciding with the zeros of the determinant $D(k, k_s)$, i.e. with the OR spectrum of resonant frequencies. Using Cauchy's residue theorem and the meromorphic property of the constructed Green's function, it can be represented as the sum of the resonant and non-resonant parts. The first part is completely determined by the solution of the spectral problem and is expressed through the eigenfunctions $g_s(k_s, x, y)$ associated with to the zeros of the determinant $D(k, k_s)$, that is, to the OR resonant frequencies. To substantiate this statement, we draw on the main sheet of the Riemann surface the contour C, which enclose the part of the frequency spectrum of interest and the point k corresponding to the source frequency, as well as the contours set, C_s, enclosing each resonant point, k_s. and k separately. Then from equations (3), (6) and Cauchy's residue theorem follows [9]:

$$E(k,x,y) = \sum_{s=1}^{N} \frac{A_s g_s(k_s,x,y)}{k-k_s} + \hat{E}(k,x,y), \quad (7)$$

where N is the number of poles enclosed by counter C_s;

$$A_s = \iint j_\omega(x_0,y_0) g_s(k_s,x_0,y_0) dx_0 dy_0;$$

$$\hat{E} = \frac{1}{2\pi i} \int\int\oint_{C_s} \frac{j_\omega(x_0,y_0) G_1(\xi,x,y,x_0,y_0) d\xi dx_0 dy_0}{\xi - k} + \int\int j_\omega(x_0,y_0) G_0(k,x,y,x_0,y_0) dx_0 dy_0 \quad (8)$$

For resonant frequencies with a small imaginary part $|Im\, k_s| \ll |Re\, k_s|$ for $k \approx Re\, k_s$ one of the first terms increases, while the rest remain limited. Due to the discreteness of the resonant frequencies the contour C_s can be always chosen so that $N = 1$. If non-resonant radiation can be neglected, then the spatial structure of the excited field does not depend on the frequency of the source and is described by the related eigenfunction of the OR. Non-resonant fields arise due to the radiation of the source directly into free space and a single (non-resonant) scattering of radiation on OR mirrors. The resulting non-resonant energy losses of the source are not covered by the concept of diffraction losses, characterized by the value of $Im\, k_s$, and in the energy balance they must be taken into account separately. At the same time, as shown in [9], with an exact resonance $k = k_s'$ the non-resonant part of the OR Green's function can be represented as:

$$\hat{G}(k) = \frac{i}{4} H_0^{(1)}(k|r - r'|) + 2\pi \frac{d\tilde{G}_1(\xi)}{d\xi}|_{\xi=k_s} \quad (9)$$

where $\tilde{G}_1(\xi) \overset{d}{=} i(\xi - k_s)G_1(\xi)$ is the Green's function regularized in vicinity of the resonance $\xi = k_s$. From (9) it can be seen that a high *Q*-factor of the resonance does not guarantee a small amount of power of non-resonant radiation. To reduce it, it is necessary to minimize the second term in (9). Derivative of the function $\tilde{G}_1$ by the spectral parameter at the point corresponding to the pole $\xi = k_s$ does not depend on the frequency of excitation source and might be considered as another "intrinsic" characteristic of the OR as the intrinsic *Q*-factor of the resonant and its eigenfunction.

4. Equations for slowly changing fields

The representations of the Green's function and the excited fields as the sum of the resonant and non-resonant terms (7) and (8) makes it possible to obtain an equation for the field excited by the RF current with a slowly changing amplitude in the vicinity of the isolated resonance. Let's introduce the scattered field function according to the formula:

$$E^{sc}(x,y) = E(x,y) - \iint G_0(k,x,y,x',y') j(k,x',y') dx'dy'$$

Then, using (7) with (9), the field $E^{sc}(x,y)$ in the vicinity of the s-th resonance can be represented as follows:

$$E^{sc}(k,x,y) \approx \frac{A_s(k) g_s(k_s,x,y)}{k-k_s} + 2\pi \iint j(k,x',y') \frac{d\tilde{G}_1(\xi,x,y,x',y')}{d\xi}|_{\xi=k_s} dx' \quad (10)$$

where $A_s(k) \overset{d}{=} \iint j(k,x',y') g_s(k_s,x',y') dx'dy'$.

This expression describes the resonant and non-resonant parts of the excited field of the stationary oscillations both inside and outside the resonant volume for any ratio of the radiation wavelength and OR dimensions. In equation (10), the non-resonant part of the excited field is written with the accuracy of the order Q_s^{-1}. It is expressed via derivative of the analytic continuation of the regularized Green's function $\tilde{G}_1$ (9) by frequency parameter taken at the point of exact resonance. If the amplitude of the RF current varies slowly in time, the source emits a frequency bandwidth localized near the carrier frequency ω . Provided that the frequency bandwidth $\Delta\omega_{max}$ is noticeably less than the distance between neighboring resonances, from (21) using a well-known technique using the inverse Fourier transform over frequency addition, it is not difficult to obtain an equation for slowly varying scattered fields:

$$\frac{dE^{sc}(t,x,y)}{dt} + i(k-k_{s)}E^{sc}(t,x,y) = \hat{C}_s(t)g_s(k_s x,y) \tag{10}$$

$$+i2\pi(k-k_s)\iint \frac{d\tilde{G}_1(\xi)}{d\xi}|_{\xi=k_s}\hat{J}(t,x',y')dx'dy'$$

where $\hat{C}_s(t)$ is slowly varying amplitude of the resonant part of the excited field;

$$\hat{J}(t,x',y') \approx \frac{2}{T}\int j_z(t',x',y')e^{i\omega t'}dt'$$

is slowly varying first harmonic of the source current density.

5. Conclusions

Based on the results of the Shestopalovs-Poedinchuk spectral theory of two-dimensional ORs with cylindrical mirrors, it is shown that the resonant (singular) part of the Green's function of those ORs is expressed via eigenfunctions of the resonant frequencies. At the same time, its non-resonant part is the sum of the Green's function for free space and the a power series in frequency detuning of the source frequency and resonant frequency of the regularized part of the Green's function. For exact resonance, $n-th$ term of this series is inversely proportional to the Q-factor of the resonant mode in the power of n, while its zero term being the frequency derivative of the analytic continuation of the regularized Green's function at the resonance point. Zero term does not depend of the source frequency and it is as much an "intrinsic" characteristic of the OR as its Q-factor and its eigen function associated with the resonance.

Using the Green's function and spectral theory of two-dimensional ORs, the equations describing stationary and slow varying fields excited in the OR by internal distributed sources with accounting the non-resonant radiation are obtained

In the author's work [9], a similar approach to solving the problem of resonant modes excitation in quasi-optical OR, described by Fredholm integral equations, was developed. In the same approximation, equations were formulated and corresponding algorithms for calculating EM fields in the OR with diffraction grating were implemented, which are suitable for describing self-oscillatory regimes in diffraction electronics devices with a non-fixed field structure and which account non-resonant radiation. In addition, within the frame of rigorous methods, the application of the obtained equations for the analysis of self-oscillations in a generator with an oscillatory system in the form of the simplest two-dimensional OR - a cylinder with a longitudinal slit is demonstrated [9].

A new form for writing the balance equations for active and reactive powers in the system "OR + Source" is proposed. This form is convenient for analyzing the non-resonant radiation and suggested the way of calculating the coefficient of radiation energy conversion of a given source into the energy of the OR resonant mode, which consists in calculating the ratio of the Q-factor of the excited oscillations to the Q-factor of the OR eigen mode [9].

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