URSI GASS 2021, Rome, Italy, 28 August - 4 September 2021

Density-Matrix Constraints in Classical Electromagnetism

Owen D. Miller*(1), Hanwen Zhang(1)
(1) Applied Physics Department and Energy Sciences Institute, Yale University, New Haven, CT, 06511

Any linear electromagnetic system whose inputs contain noise or any other incoherence must trade off maximal response for robustness. We develop a mathematical formulation of this statement, deriving upper bounds for Hermitian "observables" (i.e. response-function operators) averaged over partial or full incoherence. We show that scattering-channel density matrices, which neatly encapsulate incoherence even in classical electromagnetism, obey *majorization* bounds that reveal entropy-like constraints in passive linear systems. We show that the maximization of any (quadratic) electromagnetic response function under coherence constraints results in a semidefinite program whose solution is analytically known in terms of the eigenvalues of the relevant operators and density matrices. We apply this formalism to the design of multi-angle concentrators, generalizing the classical brightness theorem to discrete-angle applications. More broadly, our work provides a general theory of density-matrix constraints in classical electromagnetism.

If we represent the incoming waves in the incoming-channel basis with a vector $\mathbf{c}_{\text{in}}$, and the outgoing waves in the outgoing-channel basis $\mathbf{c}_{\text{out}}$, they are connected through the corresponding scattering matrix $\mathbf{S}$ in these bases:

$$\mathbf{c}_{\text{out}} = \mathbf{S}\mathbf{c}_{\text{in}}. \tag{1}$$

Then, for any source of incoherence over which averaging is denoted by angle brackets, $\langle \cdot \rangle$, one can define density matrices for the incoming and outgoing waves by,

$$\rho_{\text{in}} = \langle \mathbf{c}_{\text{in}}\mathbf{c}_{\text{in}}^{\dagger} \rangle, \quad \rho_{\text{out}} = \langle \mathbf{c}_{\text{out}}\mathbf{c}_{\text{out}}^{\dagger} \rangle. \tag{2}$$

For N_{in} incoming channels and N_{out} outgoing channels, the density matrices are $N_{\text{in}} \times N_{\text{in}}$ and $N_{\text{out}} \times N_{\text{out}}$ matrices, respectively. The diagonal terms represent incoherent power flow, while nonzero off-diagonal terms indicate coherence between channels. In the SVD framework, the channels are normalized such that $\mathbf{c}_{\text{in}}^{\dagger}\mathbf{c}_{\text{in}}$ and $\mathbf{c}_{\text{out}}^{\dagger}\mathbf{c}_{\text{out}}$ represent power flow. For normalized incoming power we have $\mathbf{c}_{\text{in}}^{\dagger}\mathbf{c}_{\text{in}} = 1$, and for a nonamplifying system one cannot have excess outgoing power, thus requiring $\mathbf{c}_{\text{out}}^{\dagger}\mathbf{c}_{\text{out}} \leq 1$.

In Ref. [1], we proved that the largest eigenvalue of ρ_{out} must be less than or equal to the largest eigenvalue of ρ_{in}, i.e. $\lambda_{\max}(\rho_{\text{out}}) \leq \lambda_{\max}(\rho_{\text{in}})$, which can be interpreted as the statement that the maximum concentration of power into a single channel is the largest power flowing in an incoherent input. Here we significantly generalize that result. We show that the sum of the N largest eigenvalues of ρ_{out} must be less than or equal to the N largest eigenvalues of ρ_{in}. If we denote a descending order on eigenvalue with the notation $\lambda^{\downarrow}$, then we can write the eigenvalue bound as

$$\sum_{i=1}^{N} \lambda_i^{\downarrow}(\rho_{\text{out}}) \leq \sum_{i=1}^{N} \lambda_i^{\downarrow}(\rho_{\text{in}}), \tag{3}$$

for any N. Equation (3) appears to be the most general one that passivity implies for an electromagnetic density matrix. We discuss its derivation, via semidefinite programming, its utility, and its implications for "purifying" electromagnetic inputs in systems with gain and/or loss.

References

[1] H. Zhang, C. W. Hsu, and O. D. Miller, "Scattering concentration bounds: brightness theorems for waves," *Optica*, **6**, 10, October 2019, pp. 1321-27, doi: 10.1364/OPTICA.6.001321.