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Frequency-Bandwidth Dependent Degrees of Freedom as a Bound of Super-Directivity

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Abstract

The Degrees of Freedom (DoF) of the field are associated to the electromagnetic field at a certain distance from a minimum surface enclosing the sources, so that the reactive field is negligible. The conventional concept of DoF is based on the limited *spatial* bandwidth of the EM fields and not related with *frequency* bandwidth of the source. This paper suggests an extension of the DoF concept with a dependency on frequency bandwidth and a process for its calculation. This is done by introducing the source frequency bandwidth as a measure of the reactive energy stored in the vicinity of a radiating surface; thus, leading to a near-field DoF as twice the bound of superdirectivity for a fixed bandwidth. The problem is treated for a spherical source region by using spherical wave harmonics, but it can be extended to arbitrary radiating surfaces.

1 Degrees of Freedom of the field

The number of Degrees of Freedom (N_{DoF}) of the field radiated by arbitrary sources contained in a given minimum sphere is the minimum number of independent scalar parameters sufficient to describe the field in a certain region of space [1]. It is equivalently the number of scalar coefficients of a wave-function expansion needed for a complete and non-redundant description of the field. This number is important to establish the numerical complexity of an electromagnetic problem [2]. When the observed region is all the space around the source, and and the observer is out of the reactive region one has $N_{DoF} = 2\lceil kr_{min}\rceil^2$, where k is the free-space wavenumber, r_{min} is the radius of the minimum sphere containing the sources and $\lceil\cdot\rceil$ denotes the smallest integer number larger or equal to the argument. This number is coincident with the number of spherical wave (SW) harmonics excited on the minimum surface that are not affected by cut-off [3].

In [4], it has been emphasized that this number is also coincident with twice the maximum directivity $D_{max}^{(ns)}$ of a large non-superdirective (non-super-reactive) antenna within the same minimum sphere, namely

$$N_{DoF} = 2D_{max}^{(ns)}, \tag{1}$$

where the factor 2 results from the two orthogonal polarizations, corresponding to the transverse electric (TE) and the transverse magnetic (TM) wave-modes. In the following, we will consider the N_{DoF} as a real, instead of an integer number, understood that the integer DoF number is the integer closest to the real DoF number. It is however observed in [4] that if one considers $N_{DoF} = 2(kr_{min})^2$ the equivalence expressed in (1) is an asymptotic concept, namely valid for large sphere in terms of the wavelength and in the far zone where the reactive field is vanished.

However, in [4] is argued that the relation in (1) can be valid even for intermediate and small radii of the minimum sphere, where the limit of maximum directivity can be found as suggested by Harrington [5]. In [5], the maximum directivity of a non-super-reactive source is obtained as a function of the maximum polar index of the spherical wave expansion. This gives $D_{max}^{(ns)} = (N_{max})^2 + 2N_{max}$ where $N_{max} \geq 1$ is the maximum polar index of the spherical harmonics. The same result can be obtained by the convex optimization procedure defined in [6]. Since the maximum polar index for non-super-reactive antennas is $N_{max} \approx kr_{min}$, the maximum directivity for non-super-reactive antennas can be defined as [4]

$$D_{max}^{(ns)} = \begin{cases} (kr_{min})^2 + 2kr_{min} & \text{for} \quad kr_{min} \geq 1.5 \\ (kr_{min})^2 + 3 & \text{for} \quad kr_{min} < 1.5 \end{cases} \tag{2}$$

Using (2) in (1) gradually holds from large radius, where $N_{DoF} = 2(kr_{min})^2$ till $kr_{min} < 1$ where we know that the degrees of freedom are 6 (three orthogonal electric and three orthogonal magnetic dipoles) and the maximum directivity is 3, namely the one of a Huygens' source. It should be observed that an alternative formula has been also suggested in [4], by counting the available spherical modes at the surface of the minimum sphere weighting them with their inverse radiation-Q.

2 DoF and Superdirectivity Bounds

The N_{DoF} defined by (1) and (2) is inherently unaffected by the reactive field close to the minimum sphere, or, in other terms, by the excitation of harmonics beyond

$N_{max} = \lceil kr_{min} \rceil$. If this is not very significant for large antennas, it becomes significant for small to medium size antennas. Therefore, a more accurate way is needed for extending the concept of DoF in the near-field region.

It is obvious that if one can excite all possible harmonics over the minimum sphere with arbitrary coefficients, the number of degrees of freedom in the near zone becomes infinite. The spatial bandwidth limitation of the Green's function gradually reduces the N_{DoF} till establishing them out of the reactive region. However, the classical concept of DoF is too restrictive concerning with the frequency bandwidth of the source.The allowance of a frequency bandwidth limitation (in practice less restrictions in the antenna requirements), increases the number of DoF. This is intuitively justified by the fact that a rich dose of reactive energy may imply an equivalent effective area of the source larger than the maximum cross-section of the minimum sphere, at the expenses of a smaller frequency bandwidth [7]. This concept is quite similar to the one of super-reactive antennas, where super-directivity (namely directivity larger than the one expressed by (2) can be obtained only at the expenses of a limited bandwidth. It is therefore reasonable to state that (1) can hold when apply a limitation on bandwidth, provided that the directivity of non-supereactive source is substituted with the one of a super-reactive source.To derive practical formulas, we will use the same process used by [5] for constructing the super-directivity bounds.

Therefore, we assume to excite harmonics with polar index up to $\bar{N}_{max}$, larger than $N_{max} = kr_{min}$. This limit is fixed by the quality factor Q of the system of currents that generates the maximum directivity. (Note that for large values, Q can be seen as the inverse of the relative bandwidth). This is given by [5] as

$$Q(\bar{N}_{max}, kr_{min}) = \frac{\sum_{n=1}^{\bar{N}_{max}} (2n+1) Q_n(kr_{min})}{\bar{N}_{max}^2 + 2\bar{N}_{max}} \quad (3)$$

where $Q_n(kr_{min})$ is the quality factor of the n-th polar harmonics, evaluated through the reactive energy outside the minimum sphere. Explicit, approximate expressions of $Q_n(kr_{min})$ are given in [5]; accurate expressions are given in [8] and [9], the latter two references providing equal results, even if apparently different. Figure 1 shows the quality factor Q in (3) with $Q_n(kr_{min})$ calculated as in [8]- [9] for different values of $\bar{N}_{max}$.

Inverting the expression $Q = Q(\bar{N}_{max}, kr_{min})$ in (3) for any fixed minimum sphere, leads to $\bar{N}_{max} = \bar{N}_{max}(Q, kr_{min})$. The value of $\bar{N}_{max}$ as a function of the minimum sphere radius for some values of Q is provided in Fig. 2. Finally, the frequency bandwidth limited DoF for a given Q can be evaluated as

$$\frac{1}{2} N_{DoF}^{(near)} = D_{max}^{(s)} = \bar{N}_{max}^2(Q, kr_{min}) + 2\bar{N}_{max}(Q, kr_{min}) \quad (4)$$

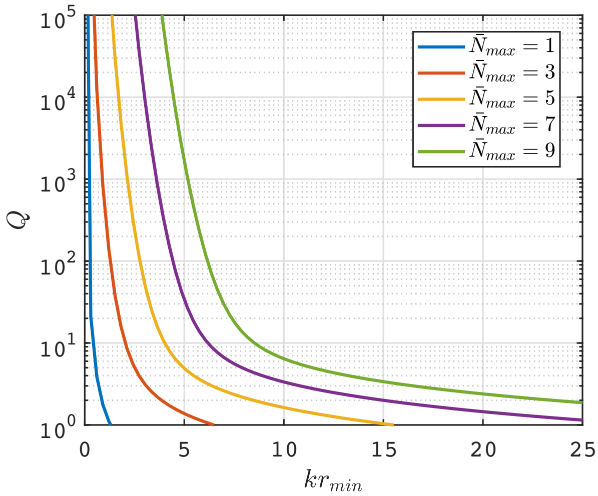

Figure 1. Q factor of the system of SW harmonics providing maximum directity as a function of the minimum sphere radius for several maximum number of harmonics $\bar{N}_{max}$ (log vertical scale).

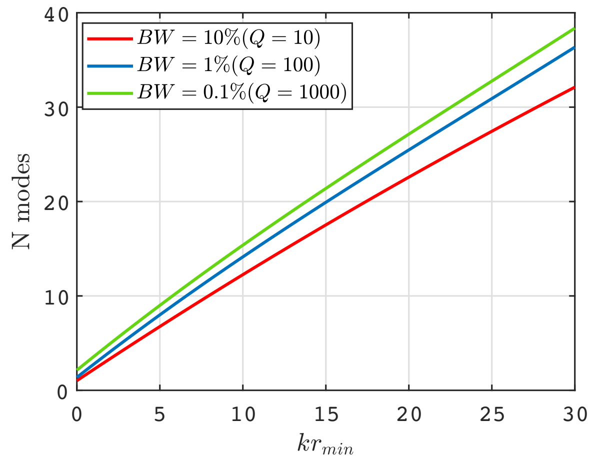

Figure 2. Maximum index of polar harmonics as a function of kr_{min} for different values of Q.

where $D_{max}^{(s)}$ is the maximum bound of superdirectivity for a certain bandwidth. We observe that this bound does not coincide with the superdirectivity limit of Harrington because of a different way we have used for calculating the energy stored in each spherical harmonic. The plot of continuous DoF and maximum super-directivity is given in Fig. 3 for different bandwidths, i.e., different values of Q factor. In this plot, we also have reported D_{max}^{ns} in (2), which corresponds to the classical bounds of non-super-reactive antennas.

3 Conclusions

The DoF in super-reactive region for a certain frequency bandwidth of the radiating currents has been introduced, and it has been defined and interpreted as half of the super-

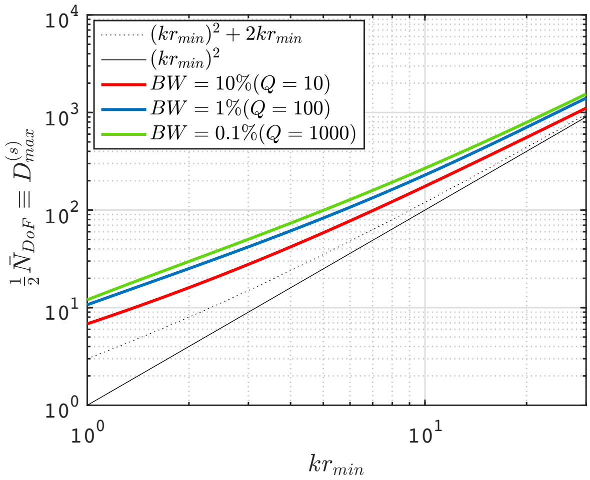

Figure 3. Number of near field DoF and bound of superdirectivity for various values of the bandwidth of the radiating current system. The staircase is $\frac{1}{2}N^{ns}_{DoF}$ and the continuous curve the is D^{s}_{max}. The dotted curve is the limit of maximum directivity for non superdirective antennas, which is obtained by $Q = 1$.

directivity bound for a given bandwidth. This contributes to estimate the computational complexity of the problem in terms of current discretization for both resonant and nonresonant problems.

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