*URSI GASS 2021, Rome, Italy, 28 August - 4 September 2021*

# Practical performance of the VEXPA estimation method in sparse regular arrays

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## Abstract

This paper presents an investigation into the practical performance of a direction of arrival (DOA) estimation method for sparse regular arrays, called VEXPA. Its ability to estimate the number of incoming signals through cluster analysis is compared to the traditional method used by dense estimation techniques. It is seen that the usefulness of the cluster analysis method depends on the underlying estimation method used by VEXPA. The effect of quantisation errors introduced by the use of an ADC is also investigated. It is found that VEXPA performs satisfactorily with one-bit data for a maximum of one signal.

## 1 Introduction

A common application of antenna array systems is to estimate the direction of arrival (DOA) of one or more incoming radio signals. Several well-established methods, such as MUSIC [1] and the Matrix-Pencil method [2] exist which traditionally use dense arrays. Such arrays adhere to the spatial Nyquist criterion that elements must be spaced less than a half-wavelength apart, so no ambiguity is caused in the output of their solutions, since aliasing does not occur.
A regular sub-sampling algorithm called VEXPA [3] has been formulated and successfully applied to the DOA estimation problem on sparse regular linear array examples using full-wave simulations in [4]. The elements in a sparse antenna array system are spaced more than a half-wavelength apart, which would normally lead to ambiguity in the output. The VEXPA algorithm overcomes this aliasing effect by intersecting the output of two sparse subarrays arranged in a co-prime configuration.
Sparse arrays have a number of advantages over dense arrays, including lower mutual coupling and an improved angular resolution. VEXPA introduces the possibility of solving the DOA problem in sparse regular arrays. It is used on top of a traditional Prony-like method, e.g. the Matrix-Pencil method, and includes features such as automatic detection of the number of incoming signals.
In this paper, the performance of VEXPA is investigated when practical non-idealities are present. Specifically, we investigate VEXPA's ability to correctly estimate the number of signals, as well as its performance when quantisation errors are introduced.

## 2 DOA Using Exponential Analysis

### 2.1 Dense Uniform Linear Array

With reference to Fig. 1, consider a uniform linear array (ULA) of $M$ antenna elements receiving $n$ signals $S_i(t)$. The output of the ULA at the $m$th element is the sum of the $n$ signals, with time delays related to $\phi_i$ and $d$, the distance between antenna elements. The narrowband signals $S_i(t)$ at

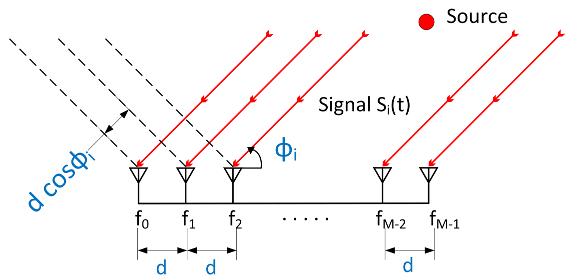


**Figure 1.** Illustration of a uniform linear array receiving a signal $S_i(t)$ at an angle $\phi_i$.

a frequency $\omega$ and time $t$ can be expressed as

$$S_i(t) = a_i(t)e^{jp_i(t)}e^{j\omega t} \tag{1}$$

with $a_i(t)$ and $p_i(t)$ denoting the slowly varying amplitude and phase of the signal, respectively. The source is assumed to be in the far-field of the antenna, so that $S_i(t)$ is a plane wave incident on the ULA. The time delay of incidence on consecutive antenna elements is given by

$$\tau_i = \frac{d\cos(\phi_i)}{c}, \tag{2}$$

with $c$ equal to the propagation velocity of the signal or in free space, the speed of light. At a time $t$ the output of the ULA at the $m$th element is

$$f_m(t) = \sum_{i=1}^{n} S_i(t + m\tau_i) \tag{3}$$

with $n$ equal to the number of signals. According to the narrowband assumption, the signal does not change noticeably as it moves across the elements of the array so that

$$\begin{aligned} f_m(t) &= \sum_{i=1}^{n} S_i(t+m\tau_i) \approx \sum_{i=1}^{n} S_i(t)\exp(j\omega m\tau_i) \\ &= \sum_{i=1}^{n} S_i(t)\exp\left(\frac{j\omega md\cos\phi_i}{c}\right). \end{aligned} \tag{4}$$

Therefore, the time delay of each signal as it moves across the array leads to a phase shift that depends on the direction of the incoming angle of the signal $\phi_i$. These angles can be recovered with the use of exponential analysis. At a fixed time $t$, the output of the ULA is called a snapshot. By introducing shorthand notations, the DOA problem is formulated as:

$$\begin{aligned} f_m &= f_m(t), & \alpha_i &= S_i(t), \\ \psi_i &= \frac{j\omega\cos\phi_i}{c}, & \Psi_i &= \exp(\psi_i d), \end{aligned} \tag{5}$$

where $f_m$ is referred to as the samples of the exponential analysis problem, $\alpha_i$ are the coefficients, $\psi_i$ the exponents and $\Psi_i$ the base terms. Considering a single snapshot, it is possible to rewrite (4) as

$$f_m = \sum_{i=1}^{n} \alpha_i \Psi_i^m, \qquad m = 0, \dots, M-1, \tag{6}$$

which can be solved from $2n$ samples if the base terms $\Psi_i$ are mutually distinct and the number of signals $n$ is known. Solutions of the base terms lead unambiguously to the directions of arrival of the signals $\phi_i$.

## 2.2 Sparse Regular Linear Array

The VEXPA algorithm requires an antenna array consisting of two sparse ULAs in a co-prime configuration, where one is a shifted version of the other, as illustrated in Fig. 2. The

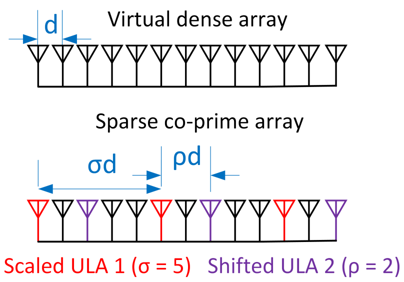


**Figure 2.** Layout of two sparse regular arrays in a co-prime configuration with an underlying virtual dense array.

array is configured from an underlying virtual dense ULA with antenna element spacing $d < \lambda/2$. The first sparse ULA is then found from a scale parameter $\sigma$ which leads to a spacing of $\sigma d$ between the elements. For sparse ULA 2, a shift parameter $\rho$ is used from which the sparse ULA 1 is shifted by a distance $\rho d$. It is required that $\sigma$ and $\rho$ be chosen as co-prime, i.e., $\gcd(\sigma, \rho) = 1$ in order to recover from the aliasing effects of each sparse ULA. The samples from sparse ULA 1 satisfy

$$f_{m\sigma} = \sum_{i=1}^{n} \alpha_i (\Psi_i^\sigma)^m, \quad m = 0, \dots, M_\sigma - 1, \tag{7}$$

from which the base terms $\Psi_i^\sigma$ can be solved using any one-dimensional Prony-like exponential analysis method. The aliasing effect caused by the sparseness of the array can be seen by the fact that all values in the set

$$\left\{ \exp\left( \psi_i d + \frac{j2\pi}{\sigma} \ell \right) \quad : \quad \ell = 0, \dots, \sigma - 1 \right\}$$

are possible solutions to $\Psi_i$. The output from the second sparse ULA satisfies

$$f_{m\sigma+\rho} = \sum_{i=1}^{n} (\alpha_i \Psi_i^\rho)(\Psi_i^\sigma)^m, \quad m = 0, \dots, M_\rho - 1, \tag{8}$$

which contains the same base terms $\Psi_i^\sigma$ as in (7). To locate the correct solution for $\Psi_i$, the coefficients $\alpha_i$ are first found from the Vandermonde structured set of linear equations using the samples $f_{m\sigma}$, after which another Vandermonde set of equations can be solved to find $\Psi_i^\rho$ using the samples in (8). From the solutions $\Psi_i^\rho$, all values in the set

$$\left\{ \exp\left( \psi_i d + \frac{j2\pi}{\rho} \ell \right) \quad : \quad \ell = 0, \dots, \rho - 1 \right\}$$

are possible solutions to $\Psi_i$. Now, since $\sigma$ and $\rho$ were chosen to be co-prime, their intersection results in the de-aliased solution of $\Psi_i$. VEXPA overestimates the number of incoming signals, and repeats the underlying estimation method multiple times using different snapshots. By doing this, the true base terms will form clusters, whereas the spurious ones will be scattered. A clustering algorithm, such as DBSCAN, is used to detect the clusters and thereby determine the number of signals.

# 3 Estimation of the number of signals

In the previous section, we described how VEXPA makes use of cluster analysis to estimate the number of incoming signals. Traditional dense DOA estimation methods use a different approach. The covariance matrix of the output array data is decomposed into the signal and noise subspaces by using either a singular value decomposition (SVD) or eigenvalue decomposition. With the number of signals denoted as $n$, it is shown in [1] that only $n$ eigenvalues rise up above the noise floor, while the others are equal to the noise variance. The number of signals can therefore be determined by subtracting the multiplicity of the smallest eigenvalue from the total number of eigenvalues.

## 3.1 Experimental Setup

The two methods for estimating the number of signals are compared by performing two simulations: one with VEXPA in its original form, and another with the cluster analysis step of VEXPA replaced with the eigenvalue-counting method. The latter method is incorporated by adding this step to the underlying estimation method, whereas the cluster analysis is performed on the base terms returned by the underlying method.
MATLAB is used to assemble the experimental setup, with ten elements in ULA 1 and five in ULA 2. This means a

maximum of $M_\sigma/2 = 5$ can be detected [4]. We vary the number of signals from one to four, with the respective angles of arrival at 90°, 70°, 60° and 5°, and the signal-to-noise ratio (SNR) is varied from 0 to 30 dB. We use both Root-MUSIC and the Matrix-Pencil method as underlying methods to VEXPA, and 100 Monte Carlo runs are performed. The scaling and shifting parameters are $\sigma = 11$ and $\rho = 5$.

### 3.2 Results

Fig. 3 shows the results from the experiment as explained above. The success rate is calculated as the percentage of the Monte Carlo runs that are successful. For a run to be considered successful, the number of signals must be estimated correctly, and the errors of the estimated angles need to be smaller than the angular resolution, given by $\Delta_\phi \approx \frac{\lambda}{D}$. $D$ is the total length of the array in the linear case.
We see a considerable difference between the results of the different underlying methods. The main difference between using Root-MUSIC and the Matrix-Pencil method is that the latter is performed by using a single snapshot of data, whereas the former uses a subset of the total snapshots in order to estimate the covariance matrix more accurately. This means that, when Root-MUSIC is used together with the cluster analysis method, a slightly less accurate covariance matrix is used, as not all snapshots are used. This explains the slight improvement in success rate when using the eigenvalue counting method.
When using the eigenvalue counting method with the Matrix-Pencil method underlying, the results seem to worsen as the number of incoming signals are increased. On the other hand, when using the cluster analysis method, the results do not seem to depend on the number of signals to such an extent. For the cluster analysis method, the overestimation of the number of signals ensures that all possible base terms are returned, and later those that do not form clusters are discarded. For the eigenvalue method, some signals are wrongly identified as noise when their corresponding eigenvalues are too small. As the number of signals increases, it is evident that the distinction between noise eigenvalues and signal eigenvalues become vaguer.
From this, we can conclude that DOA estimation methods that are solved per snapshot can benefit from the cluster analysis method, whereas for methods using the covariance matrix, the effect of a higher number of snapshots has a more significant impact, and both the eigenvalue method and cluster analysis method are accurate.

## 4 Quantisation errors

In order to capture the data successfully from the antenna ports, an analogue-to-digital converter (ADC) is required. The resolution of the ADC contributes to the accuracy thereof, but as higher-resolution ADCs are more costly, low-resolution ADCs are often preferred in antenna array applications. For example, a one-bit ADC can very easily be implemented as it consists solely of a comparator. In [6] it was shown that MUSIC performs accurately when one-bit data is used. The question therefore arises whether VEXPA is able to do the same.

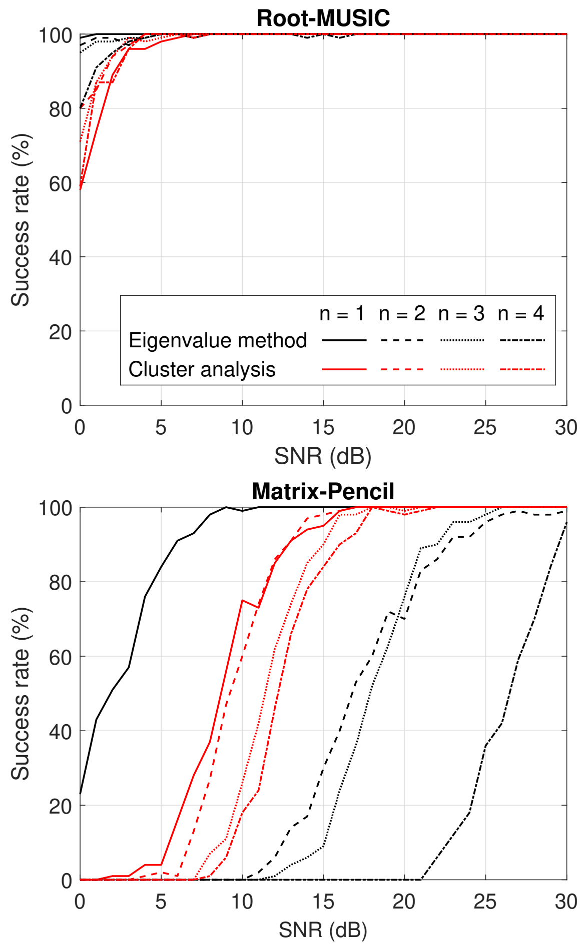


**Figure 3.** Success rate estimating the number of incoming signals by using two methods: identifying eigenvalues corresponding to the signal subspace (black), and using cluster analysis (red).

### 4.1 Experimental setup

Firstly we compare the performance of the two underlying methods on a dense configuration (12 elements, $0.48\lambda$ spacing). The number of signals vary from one to three and the angles of arrival are 10°, 30° and 90°. The number of bits of the ADC is varied from 1 to 8.
Next, we perform a similar experiment, but now we use VEXPA with a sparse co-prime configuration. The virtual dense spacing is $0.48\lambda$ and the scaling and shifting parameters are $\sigma = 11$ and $\rho = 5$. As before, 100 Monte Carlo runs are performed.

### 4.2 Results

The root-mean-square (RMS) error of Root-MUSIC and the Matrix-Pencil method with dense configurations are shown in Fig. 4. As expected from [6], we see that Root-MUSIC

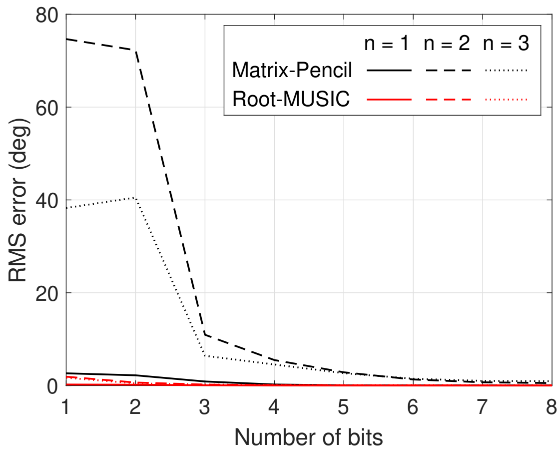


**Figure 4.** Performance of Matrix-Pencil and Root-MUSIC with a dense array configuration, with data quantised by ADCs of different bit sizes. Root-MUSIC performs accurately with low-resolution quantised data, whereas Matrix-Pencil has difficulty estimating the correct directions when multiple signals are present.

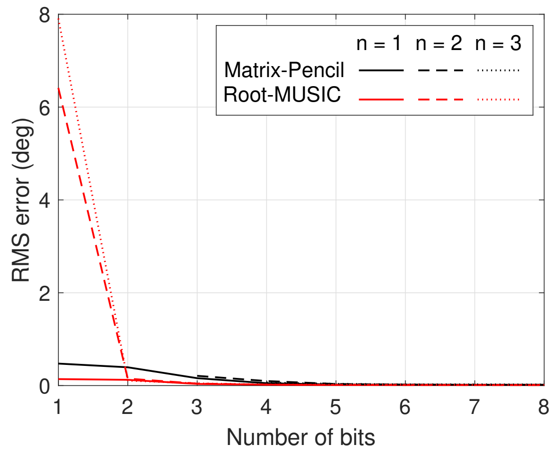


**Figure 5.** Performance of VEXPA (with Matrix-Pencil and Root-MUSIC underlying) with a sparse co-prime array configuration, with data quantised by ADCs of different bit sizes. Root-MUSIC performs accurately with low-resolution quantised data, whereas Matrix-Pencil does not return any DOA when multiple signals are present.

delivers small errors even when the number of ADC bits is as low as 1. On the other hand, the Matrix-Pencil method performs well with few bits only when a single signal is present.

In Fig. 5 we show the results of VEXPA with a sparse configuration using quantised data. For one incoming signal, the results are comparable to those of the dense setup in Fig. 4. For multiple signals, however, even Root-MUSIC cannot estimate the DOAs accurately with one-bit data. The Matrix-Pencil method fails to return any DOAs for multiple signals for data quantised by fewer than three bits.

The lowered accuracy at low-resolution data for the co-prime setup is introduced by the fact that the signal coefficients $\alpha_i$ play an important role in the formulation of the Vandermonde system in (8). As quantisation decreases the resolution of the signal amplitudes, a significant error is added to these coefficients, which negatively affects the performance of VEXPA.

We can therefore conclude that, if only low-resolution quantised data is available, the preferred underlying method for VEXPA is Root-MUSIC, although for multiple signals the performance will be compromised.

## 5 Conclusion

The investigation of the practical performance of VEXPA, a DOA estimation for sparse arrays, is presented. It is shown that, to estimate the number of signals, the cluster analysis method introduced by VEXPA is useful when incorporated with an underlying method that uses a single snapshot. When quantisation errors are considered, it is shown that for one signal, VEXPA performs comparably to its underlying method, but if multiple signals are present, the performance worsens.

## 6 Acknowledgements


This work is based on the research supported wholly by the National Research Foundation of South Africa (Grant Number: 75322).


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