URSI GASS 2021, Rome, Italy, 28 August - 4 September 2021

Wave Field in the Plasma Layer with a Linear Profile and Random Irregularities

M. V. Tinin
Irkutsk State University, 20 Gagarin blvd., Irkutsk, Russia 664003

Abstract

The previously obtained integral representation for the point source field in a layer with a linear profile and in the presence of random irregularities of different scales in the vicinity of the reflection level is studied. Asymptotic methods are used to reduce the field representation as a sevenfold integral to integrals of lower order. Conditions for the applicability of such representations are presented. Formulas and results of the numerical simulation of the frequency coherence function of a wave reflected from the turbulent plasma layer are reported.

1 Introduction

Improvement of environmental diagnostic facilities depends on the development of a sounding signal model, which in turn is determined by the level of methods of describing wave propagation in an inhomogeneous medium. A peculiarity of irregularities of both ionospheric plasma and thermonuclear fusion plasma is their multiscale nature. This is due to a wide spectrum of plasma turbulence. In addition, during vertical ionospheric sounding and thermonuclear fusion plasma sounding, it is necessary to take into account signal reflection in the presence of random irregularities.

The most visual and convenient description of wave propagation in inhomogeneous media is provided by the geometrical optics (GO) approximation. Unfortunately, this approximation cannot correctly describe effects of irregularities with the transverse size smaller than the Fresnel radius. These effects are usually described using Rytov or Born approximations. These approximations can, however, be applied only to weak fluctuations. Strong fluctuations linked to the occurrence of random caustics are described by methods of parabolic equation, phase screen, and path (Feynman) integrals. Yet, these approaches are difficult to use for studying the effect of random irregularities in the vicinity of such a caustic as the reflection point. The caustics are taken into account using integral representations obtained from the Fourier transform of the desired solution in some coordinates. As a result, the solution is sought in a mixed coordinate-phase space. This assumes a known structure of the desired field, which is determined by the location of caustics and other focal points, thus impeding the use of these methods for diagnostic problems.

In [1-3], a method of constructing integral representations by not only changing spatial coordinates to phase ones, but also by switching from a 3D (or 2D) problem to an extended problem in more dimensions has been proposed. This problem was then addressed using the Fourier transform in both source and observer coordinates. Such double weighted Fourier transform (DWFT) can describe the field in multiscale inhomogeneous media, as indicated by the transition of its results to the results of other methods in the range of their applicability. The initial version of the DWFT method [1-3] has been obtained in a small-angle approximation, when the wave equation (Helmholtz equation) can be reduced to a parabolic one. To solve the problem of large-angle scattering, including backscattering and reflection, the wave equation by the Fock proper-time method [4] was reduced to the parabolic equation without small-angle propagation conditions [5]. The DWFT solution of this equation is consistent with the GO approximation, the Born approximation, and with the strict solution of the point source problem in a medium with the linear profile of permittivity in the absence of irregularities [6].

Unfortunately, this solution takes the form of a seven-fold integral from a rapidly oscillating function [6]. Computing such an integral representation and resulting statistical moments is very difficult. In this report, we make some asymptotic calculations reducing the seven-fold integral representation to an integral of lower order. This allows us to study statistical characteristics of the signal sounding an inhomogeneous plasma.

2 Solving the wave problem for the linear profile by combining the DWFT method and the Fock method

As is known, at the time t dependence $\exp[-i\omega t]$, the solution of the wave equation reduces to the solution of the Helmholtz equation

$$\Delta G(\mathbf{r},\mathbf{r}_0)+k^2\varepsilon(\mathbf{r},f)G(\mathbf{r},\mathbf{r}_0)=\delta(\mathbf{r}-\mathbf{r}_0) \qquad (1)$$

with the condition of descending solution at $|\mathbf{r}-\mathbf{r}_0|\to\infty$. Here, $\mathbf{r}_0=\{\boldsymbol{\rho}_0,z_0\}=\{\boldsymbol{\rho}_0,0\}$ are the point source coordinates, $k=\omega/c=2\pi f/c=2\pi/\lambda$ is the wave number, f is the signal frequency, c and λ are the velocity of light and wavelength in free space, $N(\mathbf{r})$ and $\varepsilon(\mathbf{r},f)=1-80.6N(\mathbf{r})/f^2$ are the plasma electron density and permittivity. The solution of this problem is Green's function and can be used to find the field for various space-time structures of an emitted signal.

When using the Fock method [4] with the integral representation

$$G(\mathbf{r},\mathbf{r}_0) = -i/(2k)\int_0^{\infty} U(\mathbf{r},\mathbf{r}_0,\tau)\,d\tau \tag{2}$$

the solution of elliptic equation (1) reduces to the solution of the parabolic equation

$$\begin{gathered} 2ik\frac{\partial U(\mathbf{r},\mathbf{r}_0,\tau)}{\partial \tau} + \Delta U(\mathbf{r},\mathbf{r}_0,\tau) \\ +k^2\varepsilon(\mathbf{r},f)U(\mathbf{r},\mathbf{r}_0,\tau) = 0 \end{gathered} \tag{3}$$

with initial condition

$$U(\mathbf{r},\mathbf{r}_0,0) = \delta(\mathbf{r}-\mathbf{r}_0) \tag{4}$$

Solving this equation by the DWFT method, we can obtain [5] a solution that at scales of irregularity greater than the Fresnel radius goes into the GO approximation; and at weak fluctuations, into the Born approximation, including backscattering. In the small-angle approximation, i.e. at scales of irregularities greater than the wavelength, this solution gives results of the Rytov approximation for weak fluctuations and of the phase screen method when irregularities are at a large distance from a source and an observer. Moreover, in the small-angle approximation, with due regard to second-order effects, it is possible to receive results obtained earlier by the path integral method [5, 7].

Wave reflection from an inhomogeneous layer is often studied using a layer with the linear profile of the background (for an undisturbed medium) permittivity as a model of wave propagation medium. In this case, plasma permittivity is given as a sum of background $\bar{\varepsilon}(z)$ and random $\tilde{\varepsilon}(\mathbf{r})$ components

$$\varepsilon(\mathbf{r},f) = \bar{\varepsilon}(z,f) + \tilde{\varepsilon}(\mathbf{r},f)\,. \tag{5}$$

Here

$$\bar{\varepsilon}(z,f) = 1 - z/L(f) = 1 - 80{,}6\bar{N}(z)f^{-2}\,, \tag{6}$$

$$\tilde{\varepsilon}(\mathbf{r},f) = -80{,}6\tilde{N}(\mathbf{r})f^{-2}\,, \tag{7}$$

$L^{-1}(f) = 80.6L_N^{-1}f^{-2}$, $\tilde{\varepsilon}(\mathbf{r},f) = -80{,}6\tilde{N}(\mathbf{r})f^{-2}$, $\bar{N}(z) = z/L_N$ is the average plasma electron density; $\tilde{N}(\mathbf{r})$ is the random component of electron density with zero mean and the spectrum $\Phi_N(\boldsymbol{\kappa})$. For such a layer, the DWFT solution [6] can be written as follows:

$$\begin{gathered} G(\mathbf{r},\mathbf{r}_0) = C\int_0^{\infty} d\tau \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^3s\,d^3p\,\tau^{3/2} \\ \times\exp\left\{ik\left[\bar{\Phi}(\mathbf{p},\mathbf{s},\tau) + \tilde{\Phi}(\mathbf{p},\mathbf{s},\tau)\right]\right\}, \end{gathered} \tag{8}$$

$$\begin{gathered} \bar{\Phi}(\mathbf{p},\mathbf{s},\tau) = \tau\left(\bar{\varepsilon}(z) + \bar{\varepsilon}(z_0)\right)/4 \\ +(\mathbf{r}+\mathbf{r}_0)^2/(2\tau) - 2(\mathbf{ps}\tau - \mathbf{pr} + \mathbf{sr}_0) - \tau^3/\left(96L^2\right), \end{gathered} \tag{9}$$

$$C = -4k^{7/2}(2\pi)^{-9/2}\exp(-i\pi/4)\,. \tag{10}$$

Unlike [6], here (see also [8]) we use another record of phase disturbance to avoid the singularity at the turning point:

$$\begin{gathered} \tilde{\Phi}(\mathbf{p},\mathbf{s},\tau) = 0.5\int_{-\tau/2}^{\tau/2} \tilde{\varepsilon}\left[(\mathbf{s}_\perp + \mathbf{p}_\perp)\tau' + (\mathbf{s}_\perp - \mathbf{p}_\perp)\tau/2,\right. \\ \left.(s_z + p_z)\tau' + (s_z - p_z)\tau/2 - \left(\tau'^2 - \tau^2/4\right)/(4L)\right]d\tau'. \end{gathered} \tag{11}$$

Here,

$\mathbf{p} = \{\mathbf{p}_\perp, p_z\} = \{p_x, p_y, p_z\}$, $\mathbf{s} = \{\mathbf{s}_\perp, s_z\} = \{s_x, s_y, s_z\}$.

Field (8) in the absence of irregularities ($\tilde{\varepsilon} = 0$) coincides with the strict solution [6]. Therefore, this solution can be used to examine the effect of different irregularities on wave reflection from a layer. Nonetheless, multiple integration of the rapidly oscillating function impedes the use of Formula (8) in diagnostic problems. Attempts have therefore been made to reduce dimension of this representation by asymptotically calculating the integrals. For this purpose, in [6] an assumption has been made about a small contribution of irregularities with the transverse sizes less than the Fresnel radius, which are located in the vicinity of the reflection point (see also numerical simulation results in [8]). This allowed us to calculate some integrals. Here, we derive the corresponding expressions under weaker constraints.

3 Application of asymptotic integration methods

In asymptotic integration, it is important to separate fast and slow components of the integrand. This in (8) is hampered by the substantial dependence of $\tilde{\Phi}(\mathbf{p},\mathbf{s},\tau)$ on τ. Therefore, in (8) we make the following substitution of variables

$$\mathbf{s}_\perp = \mathbf{s}_{+\perp} + 0.5\mathbf{s}_{-\perp},\ \mathbf{p}_\perp = \mathbf{s}_{+\perp} - 0.5\mathbf{s}_{-\perp}, \tag{12}$$

$$s_z = s_{+z} + \frac{1}{\tau}\left(z_b - \frac{\tau^2}{16L}\right),\ \ \mathrm{p}_z = s_{+z} - \frac{1}{\tau}\left(z_b - \frac{\tau^2}{16L}\right). \tag{13}$$

As a result, we get at $z_0 = 0$

$$\begin{gathered} G(\mathbf{r},\mathbf{r}_0) = 2C\int_{-\infty}^{\infty} dz_b \int_{-\infty}^{\infty} d^2s_{-\perp} \int_{-\infty}^{\infty} d^3s_+ \int_0^{\infty} d\tau\,\tau^{1/2} \\ \times\exp\left\{ik\left[\bar{\Phi}_s(\mathbf{s}_+, z_b, \mathbf{s}_{-\perp}, \tau) + \tilde{\Phi}_s(\mathbf{s}_+, z_b, \mathbf{s}_{-\perp}, \tau)\right]\right\}, \end{gathered} \tag{14}$$

$$\begin{gathered} \bar{\Phi}_s(\mathbf{s}_+, z_b, \mathbf{s}_{-\perp}, \tau) = \frac{\tau(2L - z)}{4L} - \frac{\tau^3}{96L^2} \\ +\frac{(\mathbf{r}-\mathbf{r}_0)^2}{2\tau} - 2\tau\left[\mathbf{s}_+ - \frac{\mathbf{r}-\mathbf{r}_0}{2\tau}\right]^2 \\ +\left[2z_b - z - \frac{\tau^2}{8L}\right]^2\frac{1}{2\tau} + \left[\mathbf{s}_{-\perp} - \frac{\boldsymbol{\rho}+\boldsymbol{\rho}_0}{\tau}\right]^2\frac{\tau}{2}, \end{gathered} \tag{15}$$

$$\tilde{\Phi}_s(\mathbf{s}_+, z_b, \mathbf{s}_{-\perp}, \tau, f) = 0.5\int_{-\tau/2}^{\tau/2} \tilde{\varepsilon}\left[\bar{\boldsymbol{\rho}}(\tau_1), \bar{\mathrm{z}}(\tau_1), f\right]d\tau_1\,, \tag{16}$$

$$\begin{cases} \overline{\boldsymbol{\rho}}(\tau_1) = 2\tau_1 \mathbf{s}_{+\perp} + \mathbf{s}_{-\perp}\tau / 2_{+\perp}, \\ \overline{z}(\tau_1) = z_b - \left(\tau_1 - 4Ls_{+z}\right)^2 / (4L) + 4Ls_{+z}^2. \end{cases} \quad (17)$$

Equations (16)-(17) indicate that $\tilde{\Phi}_s$ weakly depends on τ, so we can compute the corresponding integral. To do this, by equating the derivatives of (15) to zero, find a seven-dimensional point of the stationary phase $\{\tau_c, \mathbf{s}_{+c}, z_{bc}, \mathbf{s}_{-\perp c}\}$ of the integrand in (14)

$$\tau_c = 2L[1 \pm \xi], \quad \left(\xi = \sqrt{1 - z/L}\right) \quad (18)$$

$$\begin{cases} s_{+zc} = z / (2\tau_c), \; z_{bc} = \left(z + \tau_c^2 / (8L)\right)/2, \\ \mathbf{s}_{+\perp c} = (\boldsymbol{\rho} - \boldsymbol{\rho}_0)/(2\tau_c), \; \mathbf{s}_{-\perp c} = (\boldsymbol{\rho} + \boldsymbol{\rho}_0)/\tau_c, \end{cases} \quad (19)$$

The double sign at point of stationary phase (18) corresponds to two rays coming to the observation point: before reflection (with lower sign) and after reflection (with upper sign).

By assuming scales of the background medium L and the irregularity l_ε much greater than the wavelength, expand the deterministic part of the phase $\overline{\Phi}_s(\mathbf{s}_+, z_b, \mathbf{s}_{-\perp}, \tau)$ and the argument of function $\tilde{\varepsilon}\left[\overline{\boldsymbol{\rho}}(\tau_1), \overline{z}(\tau_1)\right]$ in the vicinity of the point of stationary phase, not expanding the last function. Hence, after integrating the integral over τ, we have

$$G(\mathbf{r}, \mathbf{r}_0) = C_1 \exp\left\{ik\left[\overline{\Phi}_c\right]\right\} \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d^2 s_{-\perp} \int_{-\infty}^{\infty} d^2 s_{+\perp}$$
$$\times \mathrm{Ai}\left[\eta\sqrt{\xi}\right] \exp\left\{i\eta^2 / \left(2l_\eta^2\right) + ik\left[\tau_c\left(\mathbf{s}_{-\perp} - \mathbf{s}_{-\perp c}\right)^2 / 2 \right.\right. \quad (20)$$
$$\left.\left. - 2\tau_c\left(\mathbf{s}_{+\perp} - \mathbf{s}_{+\perp c}\right)^2 + \tilde{\Phi}_p\left(\mathbf{s}_{+\perp}, \mathbf{s}_{-\perp}, \eta\right)\right]\right\}.$$

Here $\overline{\Phi}_c = 2L\left[1 \pm \xi^3\right]/3 + (\boldsymbol{\rho} - \boldsymbol{\rho}_0)^2 (2\tau_c)^{-1}$; $C_1 = -2\exp\left[-i\pi(1 \pm 1)/4\right] k^2 \tau_c (2\pi)^{-3}$, $\tilde{\Phi}_p(\mathbf{s}_{+\perp}, \mathbf{s}_{-\perp}, \eta)$ is defined by Formula (16) when $\tau = \tau_c = 2L[1 \pm \xi] = 2L\left[1 \pm \sqrt{1 - z/L}\right]$,

$$\begin{cases} \overline{\boldsymbol{\rho}}(\tau_1) = 2\mathbf{s}_{+\perp}\tau_1 + \mathbf{s}_{-\perp}\tau_c / 2, \\ \overline{z}(\tau_1) = L + l_c \eta\sqrt{\xi} - \left(\tau_1 - 4Ls_{+zc}\right)^2 / (4L) \end{cases} \quad (21)$$

$\mathrm{Ai}\left[\eta\sqrt{\xi}\right]$ is the Airy function;

$l_\eta = (16kL)^{1/6}(1 \pm \xi)^{-1/2} = r_{fr}(1 \pm \xi)^{-1/2} / l_c$, $r_{fr} = \sqrt{k/L}$ is the Fresnel radius at a distance of L in free space;

$l_c = r_{fr}(16kL)^{-1/6}$ is the width of pre-caustic zone ($l_c < r_{fr}$ at $kL \gg 1$).

If the conditions

$$l_\varepsilon / l_c > \xi = \sqrt{1 - z/L} > (1 \pm \xi)(16kL)^{-1/3} \quad (22)$$

hold, the scale of the Airy function $\mathrm{Ai}\left[\eta\sqrt{\xi}\right]$ is smaller than that of other functions in (20). Hence, given (see [9]) that $\int_{-\infty}^{\infty} \mathrm{Ai}[\eta] d\eta = 1$, (20) yields a modification of the stationary phase method:

$$G(\mathbf{r}, \mathbf{r}_0) \approx C_1 \int_{-\infty}^{\infty} d^2 s_{-\perp} \int_{-\infty}^{\infty} d^2 s_{+\perp} \exp\left\{ik\left[\frac{\tau_c}{2}\right.\right.$$
$$\left.\left. \times\left(\mathbf{s}_{-\perp} - \mathbf{s}_{-\perp c}\right)^2 - 2\tau_c\left(\mathbf{s}_{+\perp} - \mathbf{s}_{+\perp c}\right)^2 + \tilde{\Phi}_p\left(\mathbf{s}_{+\perp}, \mathbf{s}_{-\perp}, 0\right)\right]\right\}. \quad (23)$$

Due to the excess of the Fresnel radius over the size of the pre-caustic zone at $kL \gg 1$, condition (21) is violated only in the close vicinity of the turning point.

4 Frequency Coherence Function and its Simulation

Using the resulting solutions (20), (23), we can find various statistical moments. Measurement of the frequency coherence of a sounding signal is quite often used in the inhomogeneous plasma diagnostics. To calculate it, put in (23) $\boldsymbol{\rho} = \boldsymbol{\rho}_0$, $z = z_0 = 0$, $\xi = 1$. In this case, the major contribution to integral (16) is made by the neighborhood of the turning point $\tau_1 \approx 0$. Given this, we can get

$$G(\mathbf{r}, \mathbf{r}_0) \approx -i(2\pi)^{-1} G_0 \int_{-\infty}^{\infty} d^2 v \exp\left\{iv^2/2 + ik\tilde{\Phi}_v(\mathbf{v})\right\}, \quad (24)$$

where

$$\tilde{\Phi}_v(\mathbf{v}, \tau) = 0.5 \int_{-\sqrt{L}}^{\sqrt{L}} \tilde{\varepsilon}\left[\mathbf{v} r_{fr}\sqrt{2}, L - \tau'^2\right] d\tau', \; \tau_1 = 2\tau'\sqrt{L},$$

$G_0 = i\exp\{ik4L/3\} / (16\pi L)$ is the GO approximation of the reflected wave field in the absence of irregularities.

Using (24), for the frequency coherence function of the reflected wave we obtain

$$\Gamma_\omega(f_1, f_2) = \left\langle G(f_1) G^*(f_2)\right\rangle$$
$$= G_0(f_1) G_0^*(f_2) \hat{\Gamma}_\omega(f_1, f_2), \quad (25)$$

where $\hat{\Gamma}_\omega(\omega_1, \omega_2)$ – the normalized (reduced) frequency coherence function – is

$$\hat{\Gamma}(f_1, f_2) = \frac{i}{2\pi} \int_{-\infty}^{\infty} d^2 u \exp\left\{\frac{iu^2}{2} - \frac{1}{2} D_\omega(\mathbf{u}, f_1, f_2)\right\}, \quad (26)$$

$D_\omega(\mathbf{u}, f_1, f_2)$ is the structure function defined by

$$D_\omega(\mathbf{u}, f_1, f_2) = b_0\left\{\int_{-\sqrt{L_1}}^{\sqrt{L_1}} \int_{-\sqrt{L_1}}^{\sqrt{L_1}} d\tau_1' d\tau_2' \Psi_N\left[0, \tau_1'^2 - \tau_2'^2\right]\right.$$
$$+ \int_{-\sqrt{L_2}}^{\sqrt{L_2}} \int_{-\sqrt{L_2}}^{\sqrt{L_2}} d\tau_1' d\tau_2' \Psi_N\left[0, \tau_1'^2 - \tau_2'^2\right]$$
$$-2\int_{-\sqrt{L_1}}^{\sqrt{L_1}} \int_{-\sqrt{L_2}}^{\sqrt{L_2}} d\tau_1' d\tau_2' \Psi_N\left[\mathbf{u}\sqrt{2\left|r_{frp}^2(f_1) - r_{frp}^2(f_2)\right|},\right. \quad (27)$$
$$\left.\left. L_1 - L_2 - \tau_1'^2 + \tau_2'^2\right]\right\},$$

$b_0 = (2\pi / c)^2\, 80.6 L_N$, $L_{1,2} = L(f_{1,2}) = L_N f_{1,2}^2 / 80.6$, $\Psi_N[\boldsymbol{\rho}, z]$ is the correlation function of the plasma electron density.

For the carrier frequency $f_0 = (f_1 + f_2)/2 = 60\text{GHz}$, Figure 1 presents results of the numerical simulation of the frequency coherence dependence on the relative frequency separation $X = (f_2 - f_1)/(f_2 + f_1)$ for the Gaussian spectrum $\Phi_N(\boldsymbol{\kappa}) = l_0^3 \sigma_N^2 \pi^{-3/2} \exp\left[-l_0^2 \kappa^2\right]$ of irregularities of thermonuclear fusion plasma. The scale of the linear layer $L_N = z / \bar{N}(z)$ was specified through $\bar{N}(z)$ at the point $z_c = 60cm$ where the plasma frequency was equal to 60 GHz. Statistically homogeneous irregularities with fluctuations equal to 1 % of the background plasma density at the level $z_c = 60cm$ ($\sigma_N = 0.01 \bar{N}(z_c)$) were assumed to occupy all the space between the beginning of the layer $z = 0$ and the height of the reflection.

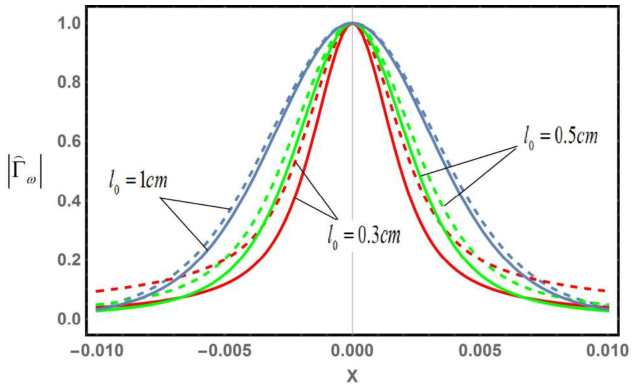

Figure 1. Reduced frequency coherence function modulus as a function of relative separation X for the Gaussian spectrum with scales of 1 cm, 0.5 cm, and 0.3 cm, calculated from DWFT formulas (solid lines) and in the GO approximation (dashed lines).

5 Conclusion

The previously obtained integral representation for the wave field reflected from a layer with random irregularities as a seven-fold integral by changing integration variables is reduced to the form such that a part of integrand weakly depends on one of the variables. This allowed us, through modification of the stationary phase method, to integrate some integrals and obtain integral representations (20), (23) of lower order. The resulting integral representations enable the study of various statistical moments in view of different (but greater than the wavelength) irregularities and their effect on wave reflection from a layer. Moreover, since the resulting solution is Green's function, it can be applied to calculations of wave scattering by various objects in inhomogeneous plasma, which can later be used to examine scattering by small-scale (less than the wavelength) irregularities. The resulting solutions are quite simple and can be adopted in the interpretation of experimental data and in its processing when studying ionospheric plasma and thermonuclear fusion plasma.

6 Acknowledgements

This study was funded by RFBR according to the research project No. 19-05-00185 and was supported by the Ministry of Education and Science of the Russian Federation (State task No. FZZE-2020-0017). I am very grateful to O.A. Kulish for her assistance in preparing the English version of the manuscript.

7 References

1. Yu. A. Kravtsov and M. V. Tinin, “Representation of wave field in a randomly inhomogeneous medium in the form of the double – weighted Fourier transform,” *Radio Sci.,* **35,** 2000, pp. 1315-1322.

2. M. V. Tinin and Yu. A. Kravtsov, “Super – Fresnel resolution of plasma inhomogeneities by electromagnetic sounding,” *Plasma Phys. Control. Fusion,* **50**, 2008, 035010 (12pp),

3. Yu. A. Kravtsov, M. V. Tinin, and S. I. Knizhin, “Diffraction tomography of inhomogeneous medium in the presence of strong phase variations,” *Journal of Communications Technology and Electronics,* **56** 2011, pp. 831–837.

4. V. Fock “Die Eigenzeit in der klassischen und in der Quantenmechanik,” *Phys. Z. Sowjetunion,* **12**, 1937, pp. 404-425 (In German).

5. M. V. Tinin, “Wave scattering in a multiscale random inhomogeneous medium,” *Waves in Random Media,* **14,** 2004, pp. 97-108.

6. M. Tinin “Wave reflection from randomly inhomogeneous ionospheric layer: 1. The method of describing the wavefield in a reflecting layer with random irregularities,” *Radio Science.* **51,** 2016, pp. 1350–1362.

7. M. V. Tinin, “Higher Approximations to Study Statistical Characteristics of Waves in Multiscale Inhomogeneous Media,” *Advances in Mathematical Physics*, **2018**, 2018, Article ID 1570407.

8. M. V. Tinin, "Role of diffraction effects in the formation of a radio signal reflected from a randomly inhomogeneous ionospheric layer," *Proc. SPIE,* **11560**, 2020, 115601Q .

9. F. W. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST Handbook of Mathematical Functions,* Cambridge University Press, 2010.