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Multi-trace, Single Source.

Kristof Cools[1]
(1) Delft University of Technology

Abstract

This contribution introduces an integral equation based domain decomposition method for the modelling of scattering of time harmonic electromagnetic waves by a penetrable obstacle. It is a single source method constructed from the *gap* idea that lies at the heart of the global multi-trace formulation. The advantages compared to the existing global multi-trace formulations are that it only requires half of the unknowns, and that the system matrix is well-conditioned without requiring further regularisation or preconditioning.

1 Introduction

Multi-trace formulations provide a clean domain decomposition approach that is amenable to fast solution techniques based on Schwarz and Calderon preconditioning [1]. These beneficial properties come at the price of having to introduce a pair of unknown fields at both sides of each interface. In this contribution, a global multi-trace formulation is introduced based on the idea underlying the class of single source integral equations (e.g. [2]): the multiplicative rather than the additive composition of the representation formulas from the interior and exterior domains. It only requires one set of unknowns (this contribution considers only electrical currents) and it has beneficial spectral properties without any further preconditioning required. The most delicate points in its practical implementation are discussed, and a conforming discretisation scheme is introduced. The method's correctness and efficiency is demonstrated by presenting numerical examples.

2 Formulation

This contribution is concerned with time-harmonic transmission problems at fixed frequency ω. Consider a device that is partitioned two domains Ω_1 and Ω_2. The unbounded complement of these domains is denoted Ω_0. The normal vector fields on Ω_1 and Ω_2 are chosen to be outward pointing. Domain Ω_i is filled by a material characterised by (ε_i, μ_i) or equivalently (κ_i, η_i).

The idea behind global multi-trace formulations is the conceptual insertion of an infinitesimally thin layer of background material at all interfaces between adjacent domains. This removes all junctions (lines along which three or more domains meet) from the setup and allows the application of proven integral equations, their discretisation, and preconditioning techniques.

The starting point for building suitable integral equations are as usual the Stratton-Chu representation theorems and the Calderon identities. Upon introduction of a gap, the boundary of Ω_0 is in fact $\partial\Omega_1 \cup \partial\Omega_2$. Unknown electric and magnetic traces $(e \times n_i, n_i \times h)$ are defined on $\partial\Omega_i$, $i = 1,2$. The representation theorem now contains contributions stemming from interactions between conceptually different but geometrically coinciding boundaries.

From an implementation point of view, the most delicate aspect about multi-trace formulations is the computation of inter-domain contributions from the double layer potential. The trace at a point of $\Gamma_1 = \partial\Omega_1$ of the double layer potential radiated by a source on $\Gamma_2 = \partial\Omega_2$ is

$$\begin{aligned} \lim_{\substack{x\to\Gamma_2 \\ x\in\Omega_0}} n_1 \times \mathscr{K}_2 j_2 &= -\lim_{\substack{x\to\Gamma_2 \\ x\in\Omega_0}} n_2 \times \mathscr{K}_2 j_2 \\ &= -\left(\frac{1}{2} - K_{12}\right) j_2 \end{aligned}$$

with

$$\begin{aligned} (\mathscr{K}_2 j)(x) &= \int_{\Gamma_2} \operatorname{curl}_x \frac{e^{-i\kappa R}}{4\pi R} j(y) dy, \quad x \notin \Gamma_2 \\ (K_{12} j)(x) &= n_1 \times pv \int_{\Gamma_2} \operatorname{curl}_x \frac{e^{-i\kappa R}}{4\pi R} j(y) dy, \quad x \in \Gamma_1 \end{aligned}$$

The limit can be interpreted as the closing of the gap between $\partial\Omega_1$ and $\partial\Omega_2$. Another consideration is that the term $1/2$ should be interpreted as the *geometric* identity, not the *topological* one:

$$\frac{1}{2} j(x) = \frac{1}{2} \int_{\Gamma_2} \delta(x-y) j(y) dy, \quad x \in \Gamma_1. \tag{1}$$

In other words, it only contributes at those $x \in \Gamma_1 \cap \Gamma_2$.

This result allows us to write the exterior Calderon identities as

$$\begin{pmatrix} K-\frac{1}{2} & -\eta T & K-\frac{1}{2} & -\eta T \\ \frac{1}{\eta}T & K-\frac{1}{2} & \frac{1}{\eta}T & K-\frac{1}{2} \\ K-\frac{1}{2} & -\eta T & K-\frac{1}{2} & -\eta T \\ \frac{1}{\eta}T & K-\frac{1}{2} & \frac{1}{\eta}T & K-\frac{1}{2} \end{pmatrix} \begin{pmatrix} e\times n_1 \\ n_1 \times h \\ e \times n_2 \\ n_2 \times h \end{pmatrix} + \begin{pmatrix} e^{inc}\times n_1 \\ n_1 \times h^{inc} \\ e^{inc} \times n_2 \\ n_2 \times h^{inc} \end{pmatrix} = 0 \quad (2)$$

The surfaces between which the operators act can be deduced from their position in the matrix. The interior Calderon identities, in the absence of sources in the interior, for regions $\Omega_{1,2}$, read:

$$\begin{pmatrix} e\times n_1 \\ n_1 \times h \\ e \times n_2 \\ n_2 \times h \end{pmatrix} = \begin{pmatrix} -K_1+\frac{1}{2} & \eta_1 T_1 & 0 & 0 \\ -\frac{1}{\eta_1}T_1 & -K_1+\frac{1}{2} & 0 & 0 \\ 0 & 0 & -K_2+\frac{1}{2} & \eta_2 T_2 \\ 0 & 0 & -\frac{1}{\eta_2}T_2 & -K_2+\frac{1}{2} \end{pmatrix} \begin{pmatrix} e\times n_1 \\ n_1 \times h \\ e \times n_2 \\ n_2 \times h \end{pmatrix} \quad (3)$$

The classic global multi-trace formulation is a generalisation of the PMCHWT equation. It is derived by subtracting the inner and outer Calderon identities.

In this contribution, a multiplicative formulation will be pursued by generalising the construction of the single source integral equation. First, an ansatz for a set of traces fulfilling the inner Calderon identities is put forward. Many options are available; here the traces of fields radiated by a source $(0, p_1, 0, p_2)$ (solely by electrical currents) is considered:

$$\begin{pmatrix} m_1 \\ j_1 \\ m_2 \\ j_2 \end{pmatrix} = \begin{pmatrix} -K_1+\frac{1}{2} & \eta_1 T_1 & 0 & 0 \\ -\frac{1}{\eta_1}T_1 & -K_1+\frac{1}{2} & 0 & 0 \\ 0 & 0 & -K_2+\frac{1}{2} & \eta_2 T_2 \\ 0 & 0 & -\frac{1}{\eta_2}T_2 & -K_2+\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ p_1 \\ 0 \\ p_2 \end{pmatrix} \quad (4)$$

The trace (m_1, j_1, m_2, j_2) is required to also fulfill the outer Calderon identities:

$$\begin{pmatrix} K-\frac{1}{2} & -\eta T & K-\frac{1}{2} & -\eta T \\ \frac{1}{\eta}T & K-\frac{1}{2} & \frac{1}{\eta}T & K-\frac{1}{2} \\ K-\frac{1}{2} & -\eta T & K-\frac{1}{2} & -\eta T \\ \frac{1}{\eta}T & K-\frac{1}{2} & \frac{1}{\eta}T & K-\frac{1}{2} \end{pmatrix} \begin{pmatrix} m_1 \\ j_1 \\ m_2 \\ j_2 \end{pmatrix} = - \begin{pmatrix} e^{inc}\times n_1 \\ n_1 \times h^{inc} \\ e^{inc} \times n_2 \\ n_2 \times h^{inc} \end{pmatrix} \quad (5)$$

At this point, a second choice is required. Only two combinations of the four equations above are required to be able to solve for (p_1, p_2). Here, the magnetic components are

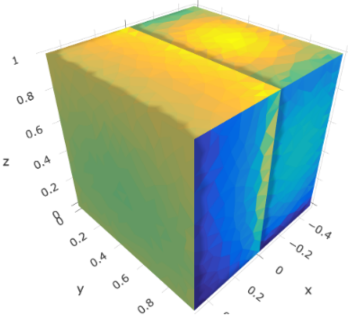

Figure 1. The left box is occupied by the background material characterised by $(\kappa = 2.0, \eta = 1.0)$. The right box is filled with a material $(\kappa' = 2.4\kappa, \eta' = \eta)$. The color scale indicates the amplitude of the tangential magnetic field.

retained:

$$\left[\frac{1}{\eta}\begin{pmatrix} T & T \\ T & T \end{pmatrix}\begin{pmatrix} \eta_1 T_1 & 0 \\ 0 & \eta_2 T_2 \end{pmatrix} - \begin{pmatrix} K-\frac{1}{2} & K-\frac{1}{2} \\ K-\frac{1}{2} & K-\frac{1}{2} \end{pmatrix}\begin{pmatrix} K_1-\frac{1}{2} & 0 \\ 0 & K_2-\frac{1}{2} \end{pmatrix}\right]\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = -\begin{pmatrix} n_1 \times h^i \\ n_2 \times h^i \end{pmatrix} \quad (6)$$

The boundaries of Γ_1 and Γ_2 are approximated by a tesselation of triangles. In this contribution, only conforming meshes are considered. This means that for each triangle in the mesh for Γ_1 that is contained in $\Gamma_1 \cap \Gamma_2$, a matching triangle in Γ_2 can be found.

To discretise the product of operators appearing in (6) both primal RWG functions f_i and dual BC functions g_i are required. The discretisation scheme can be summarised by

$$\left[\frac{1}{\eta}T_{gg}G^{gf}\eta' T_{ff} - \left(K-\frac{1}{2}\right)_{gf} G^{fg} \left(K-\frac{1}{2}\right)_{gf}\right] p^f = -(n\times h^i)_g \quad (7)$$

where $(O_{gf})_{i,j} = < n\times g_i, O f_j >$, $G^{gf} = (G_{fg})^{-1}$, and $[(n\times h^i)_g]_i = < n\times g_i, n\times h^i >$. For a detailed exposition in the case of only a single domain, the reader is referred to [3].

3 Numerical Example

Consider the two boxes from Fig. 1. The left box is occupied by background material, whilst the right box contrasts against the background: $\kappa' = 2.4\kappa$, $\eta' = \eta$. The unknowns are described by 4470 degrees of freedom. The structure is illuminated by a plane wave

$$e^i = (1,0,0)^T e^{-i\kappa x_3}. \quad (8)$$

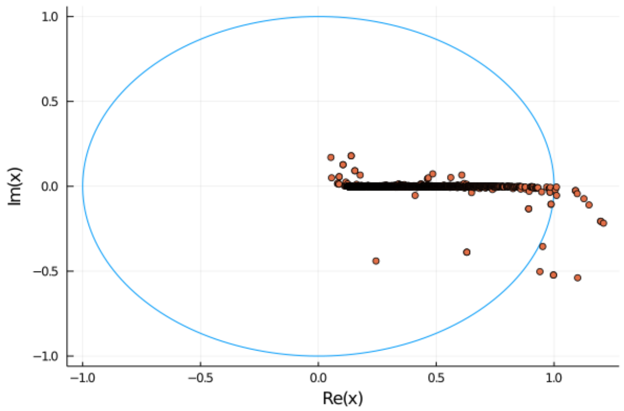

Figure 2. The eigenvalue distribution for the multi-trace single source formulation is aligned along the positive real axis and is bounded away from zero and infinity.

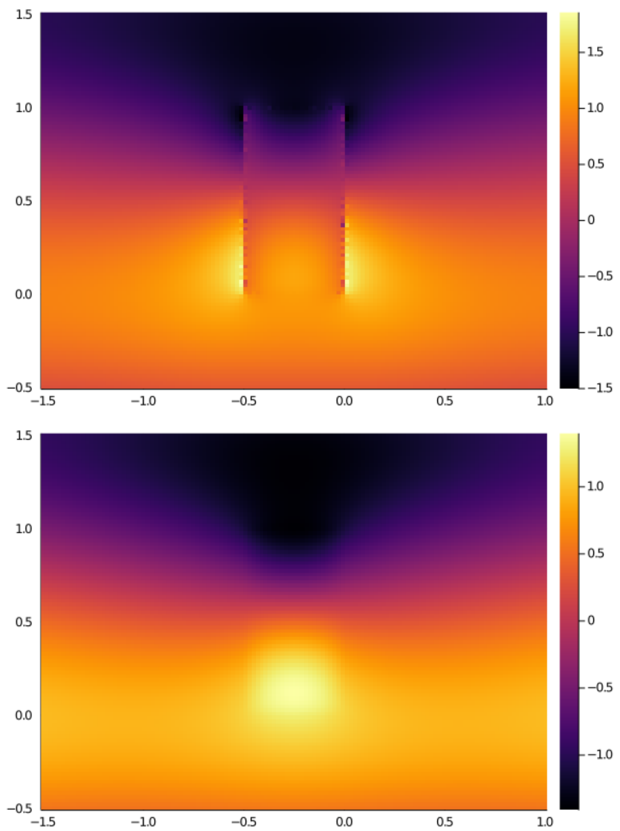

Figure 3. The e_x and h_y near field in the $y = 0.5$ section. Note that the box filled with background material (stretching from $x = -1.0$ until $x = -0.5$) is barely visible, as expected.

GMRES solution of the system up to 10^{-8} relative precision took only 45 iterations. This is is line with the qualitative features of the eigenvalue distribution in Fig. 2.

The near field (Fig. 3) computed from the solution of the multi-trace single source equation exhibits the continuity one expects. The presence of the non-contrasting box cannot be distinguished in the near field heatmaps, also as expected.

References

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