DC driven Josephson Quantum Circuits

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Abstract

Superconducting nanoelectronic devices based on the Josephson effect, and operating in the microwave frequency range have become key devices for quantum-state engineering and quantum computing, and quantum sensorics. The application of Lagrange and Hamilton methods to classical electric circuits and to circuit quantum electrodynamic circuits provides for a modeling approach. We present a quantum circuit model taking multiple DC driven Josephson junctions into account.

1 Introduction

Superconducting devices based on the Josephson effect [1, 2] and operating in the microwave frequency range have been investigated over decades [3–6]. Josephson effect based circuits and systems today are crucial for quantum-state engineering and quantum computing, quantum sensorics and quantum radar [7–12]. Early work on quantum circuits and systems has been done by Haus [13, 14] and Yurke [15]. The application of Lagrange and Hamilton methods is a powerful modeling approach for classical electric circuits as well as for circuit quantum electrodynamics [16–18]. In this contribution we discuss a quantum circuit model based on a canonical Foster equivalent circuit with multiple Josephson junctions biased from a single DC source and individually DC flux biased. This case is meaningful for DC pumped Josephson parametric amplifiers as well as for quantum circuits with synchronously operated generators, amplifiers and detectors.

2 The Josephson Junction

A Josephson tunnel junction consists of two superconductors, weakly coupled across a tunnel barrier [1–6]. For voltages $v \ll e_0\Delta$, where $e_0 = 1.602176 \times 10^{-19}$As is the electron charge and Δ is the energy gap of the superconductor the quasiparticle current can be neglected and the current $i_J(t)$ through the Josephson junction is related to the quantum phase difference φ across the junction by

$$i_J(t) = I_J \sin\varphi(t) \tag{1}$$

where I_J is the maximum Josephson current. A DC current with a maximum amplitude of I_J can flow without voltage drop across the junction. If a voltage $v(t)$ is applied, the quantum phase difference φ varies with time according to

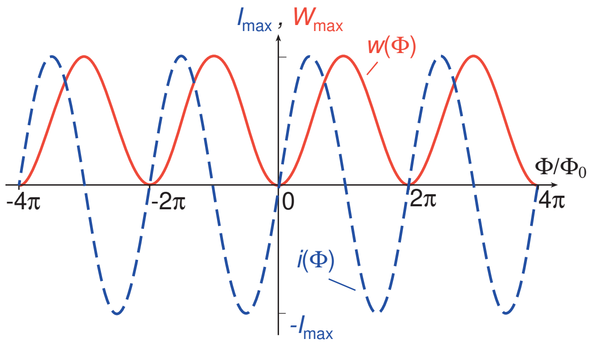

Figure 1. Josephson current and stored energy.

$$\dot{\varphi} = \frac{2e_0 v(t)}{\hbar}, \tag{2}$$

where $\hbar = h/2\pi$ is the reduced Planck constant and $h \approx 6.62606957 \times 10^{-34}\mathrm{VAs}^2$ is the Planck constant. Applying a DC voltage V_0 to the Josephson junction gives rise to sinusoidal AC current with amplitude I_J and frequency

$$f_{J0} = \frac{2e_0 V_0}{h} = 483.6\ V_0\ (\mathrm{GHz/mV})\,. \tag{3}$$

The Josephson current $i_J(t)$ is a univalent function of the integral of the voltage over time, and the Josephson junction acts as a lossless nonlinear inductor. We introduce the *flux quantum* $\phi_0 = \frac{h}{2e_0} \simeq 2.06783461 \cdot 10^{-15}$Vs. Introducing the magnetic flux $\phi(t)$ as the integral of the voltage $v(t)$ over time, i.e. $v(t) = d\phi/dt$ we obtain from (2) and (1)

$$i(t) = I_J \sin\frac{2\pi\phi(t)}{\phi_0}, \tag{4a}$$

$$v(t) = \frac{d\phi(t)}{dt}. \tag{4b}$$

The energy $w_J(t)$ stored in a Josephson junction is given by

$$w_J(\phi(t)) = W_J\left[1-\cos\frac{2\pi\phi(t)}{\phi_0}\right], \tag{5}$$

where $W_J = \phi_0 I_J/2\pi$. The energy flowing into the Josephson junction is stored in the junction, i.e. the ideal Josephson junction is non-dissipative. Figure 1 shows the dependence of the Josephson current $i(\varphi)$ and the stored energy $w_J(\varphi)$ on the quantum phase difference φ.

3 Linear Reactance Circuits

Cauer [19] and Belevitch [20] have shown that a linear reciprocal lossless multiport circuit can be represented by either a canonical Foster impedance model or a canonical Foster admittance model. Distributed linear reciprocal lossless electromagnetic circuits can be represented by canonical Foster equivalent circuits [21–23]. The canonical Foster impedance multiport representation depicted in the upper part of Fig. 2 is a series connection of elementary multiports, each of which consists of a parallel resonant circuit formed by L_ν and C_ν or a capacitor C_0 and an ideal transformer. Its impedance matrix $Z_F(\omega)$ is given by

$$Z_F(\omega) = \frac{1}{j\omega C_0} B_0 + \sum_{\nu=1}^{N} \frac{1}{j\omega C_\nu} \frac{\omega^2}{\omega^2 - \omega_\nu^2} B_\nu \qquad (6)$$

with

$$B_\nu = \begin{bmatrix} n_{\nu 1}^2 & n_{\nu 1} n_{\nu 2} & \dots & n_{\nu 1} n_{\nu M} \\ n_{\nu 2} n_{\nu 1} & n_{\nu 2}^2 & \dots & n_{\nu 2} n_{\nu M} \\ \vdots & \vdots & \ddots & \vdots \\ n_{\nu M} n_{\nu 1} & n_{\nu M} n_{\nu 2} & \dots & n_{\nu M}^2 \end{bmatrix} . \qquad (7)$$

In the following the embedding Foster circuit is represented as a Lagrangian or Hamiltonian system [24]. The Lagrangian function describing a conservative physical system is defined as the total kinetic energy minus the total potential energy of the system. The magnetic fluxes $\phi_1 \dots \phi_N$ through the inductors $L_1 \dots L_N$ are a suitable choice for the state variables, also named generalized coordinates, for the Lagrange function describing the Foster impedance circuit [16–18]. If no nonlinear circuit element is connected to the Foster circuit we consider all ports of $Z_F(\omega)$ to be open. The unperturbed Lagrange function of the Foster admittance circuit with open ports is given by

$$\mathscr{L}_0 = \sum_{\nu=1}^{N} \left(\frac{C_\nu}{2} \dot{\phi}_\nu^2 - \frac{1}{2L_\nu} \phi_\nu^2 \right) . \qquad (8)$$

To construct the Hamilton function we introduce the generalized momenta q_ν conjugated to the generalized coordinates ϕ_ν

$$q_\nu = \frac{\partial \mathscr{L}}{\partial \dot{\phi}_\nu} = C_\nu \dot{\phi}_\nu . \qquad (9)$$

Since $\dot{\phi}_\nu$ is the voltage across the inductor L_ν as well as the capacitor C_ν, the generalized momentum q_ν is the charge stored in C_ν. We obtain the Hamilton function $\mathscr{H}$ from the Lagrange function $\mathscr{L}$ by

$$\mathscr{H} = \sum_{\nu=1}^{N} q_\nu \dot{\phi}_\nu - \mathscr{L} . \qquad (10)$$

With these relations for the generalized momenta identified, the unperturbed Hamilton function of the Foster admittance circuit with short-circuited ports is

$$\mathscr{H}_0 = \sum_{\nu=1}^{N} \left(\frac{1}{2L_\nu} \phi_\nu^2 + \frac{1}{2C_\nu} q_\nu^2 \right) . \qquad (11)$$

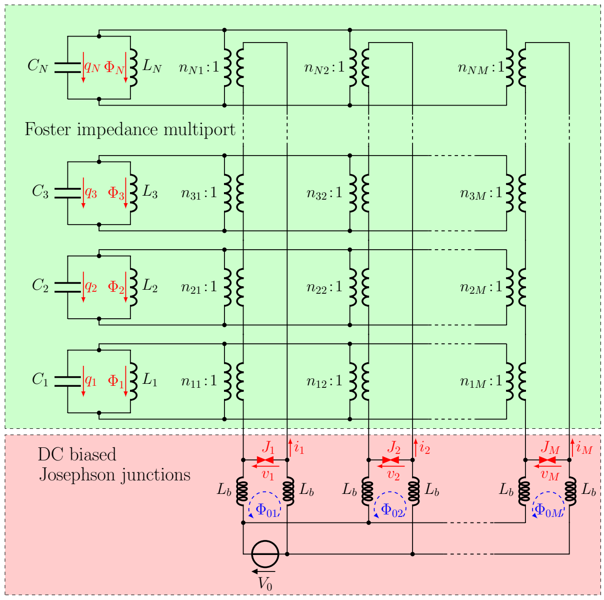

Figure 2. Foster impedance representation of DC biased Josephson junctions embedded in a linear reciprocal network.

The dynamical evolution of the Hamiltonian system is governed by the Poisson bracket [25]. For the generalized coordinates ϕ_ν, the generalized momenta q_ν and the functions $f(q_\nu, \phi_\nu, t)$ and $g(q_\nu, \phi_\nu, t)$ the Poisson bracket is given by

$$\{f, g\} = \sum_{\nu=1}^{N} \left[\frac{\partial g}{\partial q_\nu} \frac{\partial f}{\partial \phi_\nu} - \frac{\partial f}{\partial q_\nu} \frac{\partial g}{\partial \phi_\nu} \right] . \qquad (12)$$

Hamilton's equation of motion is

$$\frac{df}{dt} = \{f, \mathscr{H}\} + \frac{\partial f}{\partial t} . \qquad (13)$$

4 Circuits with Driven Josephson Junctions

We terminate the Foster impedance circuit $Z_F(\omega)$, depicted in Fig. 2, with lossless hysteresis-free nonlinear inductors $L_{N,\mu}$. The nonlinear inductors $L_{N,\mu}$ connected to the ports of $Z_F(\omega)$ are described by the nonlinear relations $g_{NL,\mu}(\phi_{N,\mu})$ between the fluxes $\phi_{N,\mu}$, the currents $i_{N,\mu}$ and the voltages $v_{N,\mu}$ as

$$i_{N,\mu} = g_{NL,\mu}(\phi_{N,\mu}) , \qquad v_{N,\mu}(t) = \frac{d\phi_{N,\mu}(t)}{dt} . \qquad (14)$$

The fluxes in the nonlinear inductors are given by

$$\phi_{N,\mu} = \phi_{N,\mu 0} - \sum_{\nu=1}^{N} n_{\nu\mu} \phi_\nu \qquad (15)$$

where $\phi_{N,\mu 0}$ are the initial fluxes of the nonlinear inductors $L_{N,\mu}$. The energy $W_{LN,\mu}(\phi_{N,\mu})$ stored in $L_{N,\mu}$ is given by

$$W_{LN,\mu}(\phi_{N,\mu}) = \int_{\phi_{N,\mu}(0)}^{\phi_{N,\mu}(t)} g_{NC,\mu}(\phi_{N,\mu}) d\phi_{N,\mu} .$$

For a hysteresis-free characteristics $g_{NL,\mu}(\phi_{N,\mu})$, the $W_{CN,\mu}(\phi_{N,\mu})$ are unique functions of $\phi_{N,\mu}$. The total Lagrange function is given by

$$\mathscr{L} = \sum_{\nu=1}^{N} \left(\frac{C_\nu}{2} \dot{\phi}_\nu^2 - \frac{1}{2L_\nu} \phi_\nu^2 \right) - \sum_{\mu=1}^{M} W_{LN,\mu}(\phi_{N,\mu}) \,. \quad (16)$$

The momenta q_ν conjugated to the coordinates ϕ_ν are

$$q_\nu = \frac{\partial \mathscr{L}}{\partial \dot{\phi}_\nu} = C_\nu \dot{\phi}_\nu \,. \quad (17)$$

This yields, using (10), the system Hamilton function

$$\mathscr{H} = \sum_{\nu=1}^{N} \left(\frac{1}{2L_\nu} \phi_\nu^2 + \frac{1}{2C_\nu} q_\nu^2 \right) + \sum_{\mu=1}^{M} W_{LN,\mu}(\phi_{N,\mu}) \,. \quad (18)$$

From this we obtain the equations of motion by inserting into (12) and (13).

We consider a circuit according to Fig. 2 where the nonlinear inductors are Josephson junctions J_μ all biased at the same DC voltage V_0. This assumption makes sense for circuits with several Josephson junctions used for generating coherent photon states or squeezed states with frequency ω_0 or ω/n with integer n or for analyzing dc pumped parametric amplifier networks with several Josephson junctions.

Let the Josephson junctions be placed in superconducting DC current loops such that the phases $\varphi_{0\mu}$ can be controlled by an applied static magnetic field. From (18) and (5) we obtain the Hamilton function

$$\begin{aligned} \mathscr{H} = & \sum_{\nu=1}^{N} \left(\frac{1}{2L_\nu} \phi_\nu^2 + \frac{1}{2C_\nu} q_\nu^2 \right) \quad (19) \\ & - \frac{\phi_0}{2\pi} \sum_{\mu=1}^{M} I_{J\mu} \cos \left[\omega_{J0} t - \varphi_{0\mu} + \frac{2\pi}{\phi_0} \sum_{\nu=1}^{N} \frac{\phi_\nu}{n_{\nu\mu}} \right] , \end{aligned}$$

Applying (13) yields the equations of motion for q_ν and ϕ_ν:

$$\frac{dq_\nu}{dt} = \frac{\partial \mathscr{H}}{\partial \phi_\nu} = \frac{\phi_\nu}{L_\nu} + \sum_{\mu=1}^{M} \frac{I_{J\mu}}{n_{\nu\mu}} \sin \left[\omega_{J0} t - \varphi_{0\mu} + \frac{2\pi}{\phi_0} \sum_{\nu=1}^{N} \frac{\phi_\nu}{n_{\nu\mu}} \right] \quad (20a)$$

$$\frac{d\phi_\nu}{dt} = -\frac{\partial \mathscr{H}}{\partial q_\nu} = -\frac{q_\nu}{C_\nu} \,. \quad (20b)$$

5 The Quantization of Reactance Networks with Driven Josephson Junctions

We consider the Foster impedance circuit which is relevant for the embedding of Josephson junctions. By analogy to (11) we find the unperturbed Hamilton operator

$$H_0 = \sum_{\nu=1}^{N} \left(\frac{1}{2L_\nu} \phi_\nu^2 + \frac{1}{2C_\nu} q_\nu^2 \right) \quad (21)$$

by substituting the flux and charge variables ϕ_ν and q_ν with the quantum mechanical operators ϕ_ν and q_ν. The flux operators ϕ_μ as the generalized position operators and the charge operators q_ν as the generalized momentum operators must fulfill the commutator relation

$$[\phi_\mu, q_\nu] \equiv \phi_\mu q_\nu - q_\nu \phi_\mu = \jmath \hbar \delta_{\mu\nu} \,, \quad (22)$$

which substitutes in the quantum regime the classical Poisson brackets (12) [26]. Hamilton's equation of motion (13) is replaced by the Heisenberg equation of motion, see [26], describing the time evolution of an operator $A(t)$ by

$$\jmath \hbar \frac{dA(t)}{dt} = [A(t), H] + \frac{\partial A(t)}{\partial t} \,. \quad (23)$$

Introducing the destruction operators a_ν and the creation operators $a_\nu^\dagger$ by

$$a_\nu = \sqrt{\frac{1}{2\hbar\omega_\nu L_\nu}} \phi_\nu + \jmath \sqrt{\frac{\omega_\nu L_\nu}{2\hbar}} q_\nu \,, \quad (24a)$$

$$a_\nu^\dagger = \sqrt{\frac{1}{2\hbar\omega_\nu L_\nu}} \phi_\nu - \jmath \sqrt{\frac{\omega_\nu L_\nu}{2\hbar}} q_\nu \,, \quad (24b)$$

where † denotes the Hermitian conjugate, we can write the Hamiltonian in the form

$$H_0 = \frac{1}{2} \sum_{\nu=1}^{N} \hbar\omega_\nu \left(a_\nu^\dagger a_\nu + a_\nu a_\nu^\dagger \right) \,, \quad (25)$$

describing a system of uncoupled harmonic oscillators, see e.g. [26]. We now consider the quantum mechanical system of nonlinear inductors connected to a Foster impedance multiport according to Fig. 2. In analogy to (18), by substituting in $W_{LN,\mu}(\phi_{N,\mu})$ the variable $\phi_{N,\mu}$ by the operator $\phi_{N,\mu}$, we obtain the Hamilton operator

$$H = H_0 + \sum_{\mu=1}^{M} H_{NL,\mu}(\phi_{N,\mu}) \,, \quad (26)$$

where H_0 is given by (25). From (15) we obtain

$$\phi_{N,\mu} = \phi_{N,\mu 0} - \sum_{\nu=1}^{N} n_{\nu\mu} \sqrt{\hbar\omega_\nu L_\nu/2} \left(a_\nu + a_\nu^\dagger \right) \,. \quad (27)$$

Let us now treat the circuit with the DC driven Josephson junction (Fig. 2) quantum mechanically. Writing the Hamilton operator H of the system with an unperturbed part H_0 and a perturbation H_1 as

$$H = H_0 + H_1 \,, \quad (28)$$

where H_0 describes the photon states in the resonant circuits $L_\nu C_\nu$ and H_1 describes the set of DC driven Josephson junctions. With (19) we obtain

$$H_1 = -\sum_{\mu=1}^{M} W_{J\mu} \cos \left[\omega_{J0} t - \varphi_{0\mu} + \sum_{\nu=1}^{N} \frac{\kappa_\nu}{n_{\nu\mu}} \left(a_\nu + a_\nu^\dagger \right) \right] \quad (29)$$

with

$$\kappa_\nu = \sqrt{\hbar\omega_\nu L_\nu/2} = \sqrt{\hbar Z_\nu/2} \,, W_{J\mu} = \frac{\phi_0 I_{J\mu}}{2\pi} \,, Z_\nu = \sqrt{\frac{L_\nu}{C_\nu}} \,,$$

where the impedance Z_ν characterizes the ν^{th} resonant circuit. Expanding (29) for small a_ν and $a_\nu^\dagger$ into a Taylor series up to second order we obtain

$$\begin{aligned} H_1 = &-\sum_{\mu=1}^{M} W_{J\mu}\cos\left[\omega_{J0}t-\varphi_{0\mu}\right] \qquad (30)\\ &+\sum_{\mu=1}^{M} W_{J\mu}\sin\left(\omega_{J0}t-\varphi_{0\mu}\right)\left[\sum_{\nu=1}^{N}\frac{\kappa_\nu}{n_{\nu\mu}}\left(a_\nu+a_\nu^\dagger\right)\right]\\ &+\sum_{\mu=1}^{M} W_{J\mu}\cos\left(\omega_{J0}t-\varphi_{0\mu}\right)\times\\ &\times\left[\sum_{\nu=1}^{N}\sum_{\nu'=1}^{N}\frac{\kappa_\nu\kappa'_\nu}{n_{\nu\mu}n_{\nu'\mu}}\left(a_\nu+a_\nu^\dagger\right)\left(a'_\nu+a'^\dagger_\nu\right)\right]. \end{aligned}$$

We consider the example of the negative resistance parametric amplifier described in [16], where $\omega_0=\omega_1+\omega_2$. For small amplitudes of a_1 we approximate the perturbation Hamiltonian H_1 by

$$H_1^{NR}=\gamma_{12}\left[a_1^\dagger a_2^\dagger e^{-j\omega_0 t}+a_1a_2e^{j\omega_0 t}\right].$$

Assuming in the initial state of the parametric amplifier a signal of amplitude w plus Gaussian noise in the signal circuit L_1C_1 and Gaussian noise only in the idler circuit L_2C_2, [6] have computed the time dependence of energy expectation value

$$\begin{aligned}\langle E(t)\rangle = \hbar\omega|w|^2\cosh^2\gamma_{12}t+\frac{\hbar\omega_1}{2}\coth\frac{\hbar\omega_1}{k_BT}\cosh^2\gamma_{12}t\\ +\frac{\hbar\omega_1}{2}\coth\frac{\hbar\omega_2}{k_BT}\sinh^2\gamma_{12}t\,. \qquad (31)\end{aligned}$$

The first term on the right-hand side is the amplified signal, the second term is the amplified noise of the signal circuit L_1C_1, and the third term is the down-converted amplified noise from the idler circuit L_2C_2.

6 Conclusion and Outlook

We have discussed the application of Lagrange and Hamilton methods to quantum circuits consisting of several synchronously DC driven Josephson junctions embedded into linear lossless reciprocal circuits. Further work will consider also dissipation and fluctuation [27] investigations including also unbiased Josephson junctions and Josephson junctions biased at different DC voltages.

7 Acknowledgement

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