DC driven Josephson Quantum Circuits

Johannes A. Russer, Michael Haider, Christian Jirauschek, and Peter Russer
Department of Electrical and Computer Engineering of the Technical University of Munich,
Arcisstr. 21, 80333 Munich, Germany; e-mails: russer@tum.de, jrusser@tum.de

Abstract

Superconducting nanoelectronic devices based on the Josephson effect, and operating in the microwave frequency range have become key devices for quantum-state engineering and quantum computing, and quantum sensorics. The application of Lagrange and Hamilton methods to classical electric circuits and to circuit quantum electrodynamic circuits provides for a modeling approach. We present a quantum circuit model taking multiple DC driven Josephson junctions into account.

1 Introduction

Superconducting devices based on the Josephson effect [1, 2] and operating in the microwave frequency range have been investigated over decades [3–6]. Josephson effect based circuits and systems today are crucial for quantum-state engineering and quantum computing, quantum sensorics and quantum radar [7–12]. Early work on quantum circuits and systems has been done by Haus [13, 14] and Yurke [15]. The application of Lagrange and Hamilton methods is a powerful modeling approach for classical electric circuits as well as for circuit quantum electrodynamics [16–18]. In this contribution we discuss a quantum circuit model based on a canonical Foster equivalent circuit with multiple Josephson junctions biased from a single DC source and individually DC flux biased. This case is meaningful for DC pumped Josephson parametric amplifiers as well as for quantum circuits with synchronously operated generators, amplifiers and detectors.

2 The Josephson Junction

A Josephson tunnel junction consists of two superconductors, weakly coupled across a tunnel barrier [1–6]. For voltages \( v \ll e_0 \Delta \), where \( e_0 = 1.602176 \times 10^{-19} \text{As} \) is the electron charge and \( \Delta \) is the energy gap of the superconductor the quasiparticle current can be neglected and the current \( i_J(t) \) through the Josephson junction is related to the quantum phase difference \( \phi \) across the junction by

\[
i_J(t) = I_J \sin \phi(t)
\]

where \( I_J \) is the maximum Josephson current. A DC current with a maximum amplitude of \( I_J \) can flow without voltage drop across the junction. If a voltage \( v(t) \) is applied, the quantum phase difference \( \phi \) varies with time according to

\[
\dot{\phi} = \frac{2e_0 v(t)}{\hbar},
\]

where \( \hbar = h/2\pi \) is the reduced Planck constant and \( h \approx 6.62607015 \times 10^{-34} \text{Vs} \) is the Planck constant. Applying a DC voltage \( V_0 \) to the Josephson junction gives rise to sinusoidal AC current with amplitude \( I_J \) and frequency

\[
f_f = \frac{2e_0 V_0}{\hbar} = 483.6 \text{ } V_0 \text{ } (\text{GHz/mV}).
\]

The Josephson current \( i_J(t) \) is a univalent function of the integral of the voltage over time, and the Josephson junction acts as a lossless nonlinear inductor. We introduce the flux quantum \( \Phi_0 = \frac{h}{2e_0} \approx 2.06783461 \times 10^{-15} \text{Vs} \). Introducing the magnetic flux \( \phi(t) \) as the integral of the voltage \( v(t) \) over time, i.e. \( v(t) = d\phi(t)/dt \) we obtain from (2) and (1)

\[
i_J(t) = I_J \sin \frac{2\pi \phi(t)}{\Phi_0}, \quad (4a)
\]

\[
v(t) = \frac{d\phi(t)}{dt}. \quad (4b)
\]

The energy \( W_J(\phi(t)) \) stored in a Josephson junction is given by

\[
W_J(\phi(t)) = W_J \left[ 1 - \cos \frac{2\pi \phi(t)}{\Phi_0} \right], \quad (5)
\]

where \( W_J = \Phi_0 I_J/2\pi \). The energy flowing into the Josephson junction is stored in the junction, i.e. the ideal Josephson junction is non-dissipative. Figure 1 shows the dependence of the Josephson current \( i(\phi) \) and the stored energy \( W_J(\phi) \) on the quantum phase difference \( \phi \).

Figure 1. Josephson current and stored energy.
3 Linear Reactance Circuits

Cauer [19] and Belevitch [20] have shown that a linear reciprocal lossless multiport circuit can be represented by either a canonical Foster impedance model or a canonical Foster admittance model. Distributed linear reciprocal lossless electromagnetic circuits can be represented by canonical Foster equivalent circuits [21–23]. The canonical Foster impedance multiport representation depicted in the upper part of Fig. 2 is a series connection of elementary multiports, each of which consists of a parallel resonant circuit formed by $L_v$ and $C_v$ or a capacitor $C_0$ and an ideal transformer. Its impedance matrix $Z_F(\omega)$ is given by

$$Z_F(\omega) = \frac{1}{j\omega C_0} B_0 + \sum_{v=1}^{N} \frac{1}{j\omega C_v} \frac{\omega^2}{\omega^2 - \omega_v^2} B_v$$

with

$$B_v = \begin{bmatrix}
    n_{v1} & n_{v1} n_{v2} & \ldots & n_{v1} n_{vM} \\
    n_{v2} n_{v1} & n_{v2} & \ldots & n_{v2} n_{vM} \\
    \vdots & \vdots & \ddots & \vdots \\
    n_{vM} n_{v1} & n_{vM} n_{v2} & \ldots & n_{vM} n_{vM}
\end{bmatrix}.$$  

(7)

In the following the embedding Foster circuit is represented as a Lagrangian or Hamiltonian system [24]. The Lagrangian function describing a conservative physical system is defined as the total kinetic energy minus the total potential energy of the system. The magnetic fluxes $\phi_1 \ldots \phi_N$ through the inductors $L_1 \ldots L_N$ are a suitable choice for the state variables, also named generalized coordinates, for the Lagrange function describing the Foster impedance circuit [16–18]. If no nonlinear circuit element is connected to the Foster circuit we consider all ports of $Z_F(\omega)$ to be open. The unperturbed Lagrange function of the Foster admittance circuit with open ports is given by

$$\mathcal{L}_0 = \sum_{v=1}^{N} \left( \frac{C_v}{2} \phi_v^2 - \frac{1}{2L_v} \phi_v^2 \right).$$  

(8)

To construct the Hamilton function we introduce the generalized momenta $q_v$ conjugated to the generalized coordinates $\phi_v$

$$q_v = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_v} = C_v \phi_v.$$  

(9)

Since $\dot{\phi}_v$ is the voltage across the inductor $L_v$ as well as the capacitor $C_v$, the generalized momentum $q_v$ is the charge stored in $C_v$. We obtain the Hamilton function $\mathcal{H}$ from the Lagrange function $\mathcal{L}$ by

$$\mathcal{H} = \sum_{v=1}^{N} q_v \dot{\phi}_v - \mathcal{L}.$$  

(10)

With these relations for the generalized momenta identified, the unperturbed Hamilton function of the Foster admittance circuit with short-circuited ports is

$$\mathcal{H}_0 = \sum_{v=1}^{N} \left( \frac{1}{2L_v} \phi_v^2 + \frac{1}{2C_v} \phi_v^2 \right).$$  

(11)

The dynamical evolution of the Hamiltonian system is governed by the Poisson bracket [25]. For the generalized coordinates $\phi_v$, the generalized momenta $q_v$ and the functions $f(q_v, \phi_v, t)$ and $g(q_v, \phi_v, t)$ the Poisson bracket is given by

$$\{f, g\} = \sum_{v=1}^{N} \left[ \frac{\partial g}{\partial q_v} \frac{\partial f}{\partial \phi_v} - \frac{\partial f}{\partial q_v} \frac{\partial g}{\partial \phi_v} \right].$$  

(12)

Hamilton’s equation of motion is

$$\frac{df}{dt} = \{f, \mathcal{H}\} = \frac{df}{dt}.$$  

(13)

4 Circuits with Driven Josephson Junctions

We terminate the Foster impedance circuit $Z_F(\omega)$, depicted in Fig. 2, with lossless hysteresis-free nonlinear inductors $L_{N,\mu}$. The nonlinear inductors $L_{N,\mu}$ connected to the ports of $Z_F(\omega)$ are described by the nonlinear relations $g_{NL,\mu}(\phi_{N,\mu})$ between the fluxes $\phi_{N,\mu}$, the currents $i_{N,\mu}$ and the voltages $v_{N,\mu}$ as

$$i_{N,\mu} = g_{NL,\mu}(\phi_{N,\mu}), \quad v_{N,\mu}(t) = \frac{d\phi_{N,\mu}(t)}{dt}.$$  

(14)

The fluxes in the nonlinear inductors are given by

$$\phi_{N,\mu} = \phi_{N,\mu}(0) - \sum_{v=1}^{N} n_v \phi_v$$  

(15)

where $\phi_{N,\mu}(0)$ are the initial fluxes of the nonlinear inductors $L_{N,\mu}$. The energy $W_{NL,\mu}(\phi_{N,\mu})$ stored in $L_{N,\mu}$ is given by

$$W_{NL,\mu}(\phi_{N,\mu}) = \int_{\phi_{N,\mu}(0)}^{\phi_{N,\mu}(t)} g_{NC,\mu}(\phi_{N,\mu}) d\phi_{N,\mu}.$$  

\[
\text{Figure 2. Foster impedance representation of DC biased Josephson junctions embedded in a linear reciprocal network.}
\]
For a hysteresis-free characteristics $g_{NL,\mu}(\phi_{N,\mu})$, the $W_{CN,\mu}(\phi_{N,\mu})$ are unique functions of $\phi_{N,\mu}$. The total Lagrange function is given by

$$\mathcal{L} = \sum_{v=1}^{N} \left( \frac{C_v}{2} \dot{\phi}_v^2 - \frac{1}{2L_v} \dot{\phi}_v^2 \right) - \sum_{\mu=1}^{M} W_{LN,\mu}(\phi_{N,\mu}). \quad (16)$$

The momenta $q_v$ conjugated to the coordinates $\phi_v$ are

$$q_v = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_v} = C_v \phi_v. \quad (17)$$

This yields, using (10), the system Hamilton function

$$\mathcal{H} = \sum_{v=1}^{N} \left( \frac{1}{2L_v} \phi_v^2 + \frac{1}{2C_v} q_v^2 \right) + \sum_{\mu=1}^{M} W_{LN,\mu}(\phi_{N,\mu}). \quad (18)$$

From this we obtain the equations of motion by inserting into (12) and (13).

We consider a circuit according to Fig. 2 where the nonlinear inductors are Josephson junctions $J_\mu$ all biased at the same DC voltage $V_0$. This assumption makes sense for circuits with several Josephson junctions used for generating coherent photon states or squeezed states with frequency $\omega_0$ or $\omega_0/n$ with integer $n$ or for analyzing dc pumped parametric amplifiers with several Josephson junctions.

Let the Josephson junctions be placed in superconducting DC current loops such that the phases $\phi_{N,\mu}$ can be controlled by an applied static magnetic field. From (18) and (5) we obtain the Hamilton function

$$\mathcal{H} = \sum_{v=1}^{N} \left( \frac{1}{2L_v} \phi_v^2 + \frac{1}{2C_v} q_v^2 \right) - \frac{\phi_0}{2\pi} \sum_{\mu=1}^{M} J_\mu \cos \left( \omega_0 t - \phi_{0,\mu} + \frac{2\pi}{\phi_0} \sum_{v=1}^{N} \phi_v \right). \quad (19)$$

Applying (13) yields the equations of motion for $q_v$ and $\phi_v$:

$$\frac{dq_v}{dt} = \frac{\partial \mathcal{H}}{\partial \phi_v} + \frac{2\pi}{\phi_0} \sum_{v=1}^{N} \phi_v \left[ \omega_0 t - \phi_{0,\mu} + \frac{2\pi}{\phi_0} \sum_{v=1}^{N} \phi_v \right], \quad (20a)$$

$$\frac{d\phi_v}{dt} = -\frac{\partial \mathcal{H}}{\partial q_v} = -\frac{q_v}{C_v}. \quad (20b)$$

5 The Quantization of Reactance Networks with Driven Josephson Junctions

We consider the Foster impedance circuit which is relevant for the embedding of Josephson junctions. By analogy to (11) we find the unperturbed Hamilton operator

$$H_0 = \sum_{v=1}^{N} \left( \frac{1}{2L_v} \phi_v^2 + \frac{1}{2C_v} q_v^2 \right) \quad (21)$$

by substituting the flux and charge variables $\phi_v$ and $q_v$ with the quantum mechanical operators $\hat{\phi}_v$ and $\hat{q}_v$. The flux operators $\hat{\phi}_\mu$ as the generalized position operators and the charge operators $\hat{q}_\mu$ as the generalized momentum operators must fulfill the commutator relation

$$[\hat{\phi}_\mu, \hat{q}_\nu] \equiv \hat{\phi}_\mu \hat{q}_\nu - \hat{q}_\nu \hat{\phi}_\mu = i\hbar \delta_{\mu,\nu}, \quad (22)$$

which substitutes in the quantum regime the classical Poisson brackets (12) [26]. Hamilton’s equation of motion (13) is replaced by the Heisenberg equation of motion, see [26], describing the time evolution of an operator $A(t)$ by

$$i\hbar \frac{dA(t)}{dt} = [A(t), \mathcal{H}] + \frac{\partial A(t)}{\partial t}. \quad (23)$$

Introducing the destruction operators $a_\nu$ and the creation operators $a_\nu^\dagger$ by

$$a_\nu = \sqrt{\frac{1}{2\hbar \omega_\nu L_v}} \phi_\nu + j \sqrt{\frac{\omega_\nu L_v}{2\hbar}} q_\nu, \quad (24a)$$

$$a_\nu^\dagger = \sqrt{\frac{1}{2\hbar \omega_\nu L_v}} \phi_\nu - j \sqrt{\frac{\omega_\nu L_v}{2\hbar}} q_\nu, \quad (24b)$$

where $^\dagger$ denotes the Hermitian conjugate, we can write the Hamiltonian in the form

$$H_0 = \frac{1}{2} \sum_{\nu=1}^{N} \hbar \omega_\nu \left( a_\nu^\dagger a_\nu + a_\nu a_\nu^\dagger \right), \quad (25)$$

describing a system of uncoupled harmonic oscillators, see e.g. [26]. We now consider the quantum mechanical system of nonlinear inductors connected to a Foster impedance multiport according to Fig. 2. In analogy to (18), by substituting in $W_{LN,\mu}(\phi_{N,\mu})$ the variable $\phi_{N,\mu}$ by the operator $\hat{\phi}_{N,\mu}$, we obtain the Hamilton operator

$$H = H_0 + \sum_{\mu=1}^{M} H_{NL,\mu}(\phi_{N,\mu}), \quad (26)$$

where $H_0$ is given by (25). From (15) we obtain

$$\phi_{N,\mu} = \phi_{N,\mu}^0 + \sum_{\nu=1}^{N} n_{\nu,\mu} \sqrt{\hbar \omega_\nu L_v/2} \left( a_\nu + a_\nu^\dagger \right). \quad (27)$$

Let us now treat the circuit with the DC driven Josephson junction (Fig. 2) quantum mechanically. Writing the Hamilton operator $H$ of the system with an unperturbed part $H_0$ and a perturbation $H_1$ as

$$H = H_0 + H_1, \quad (28)$$

where $H_0$ describes the photon states in the resonant circuits $L_vC_v$ and $H_1$ describes the set of DC driven Josephson junctions. With (19) we obtain

$$H_1 = -\sum_{\mu=1}^{M} W_{\nu,\mu} \cos \left( \omega_0 t - \phi_{0,\mu} + \sum_{\nu=1}^{N} n_{\nu,\mu} \frac{\kappa_\nu}{\sqrt{\kappa_\nu}} (a_\nu + a_\nu^\dagger) \right) \quad (29)$$

with

$$\kappa_\nu = \sqrt{\hbar \omega_\nu L_v/2} = \sqrt{\hbar Z_v/2}, \quad W_{\nu,\mu} = \frac{\phi_0 J_\mu}{2\pi}, \quad Z_v = \sqrt{\frac{L_v}{C_v}}.$$
where the impedance $Z_v$ characterizes the $v^{th}$ resonant circuit. Expanding (29) for small $a_v$ and $a_v'$ into a Taylor series up to second order we obtain

$$H_1 = -\sum_{\mu=1}^{M} W_{\mu} \cos \left( \omega_{\mu} t - \varphi_{\mu} \right)$$

$$+ \sum_{\mu=1}^{M} W_{\mu} \sin \left( \omega_{\mu} t - \varphi_{\mu} \right) \left[ \sum_{\nu=1}^{N} \frac{K_{\nu}}{n_{\nu \mu}} \left( a_v + a_v' \right) \right]$$

$$+ \sum_{\mu=1}^{M} W_{\mu} \cos \left( \omega_{\mu} t - \varphi_{\mu} \right) \times$$

$$\times \left[ \sum_{\nu=1}^{N} \sum_{\nu' = 1}^{N} \frac{K_{\nu} K_{\nu'}}{n_{\nu \mu} n_{\nu' \mu}} \left( a_v + a_v' \right) \left( a_v + a_v' \right) \right].$$

We consider the example of the negative resistance parameter amplifier described in [16], where $\omega_0 = \omega + \omega_2$. For small amplitudes of $a_1$ we approximate the perturbation Hamiltonian $H_1$ by

$$H_1^{NR} = \gamma_2 \left[ a_1^2 a_2 e^{-j\omega_0 t} + a_1 a_2 e^{j\omega_0 t} \right].$$

Assuming in the initial state of the parametric amplifier a signal of amplitude $w$ plus Gaussian noise in the signal circuit $L_1C_1$ and Gaussian noise only in the idler circuit $L_2C_2$, [6] have computed the time dependence of energy expectation value

$$\langle E(t) \rangle = \hbar |w|^2 \cos^2 \gamma_2 t + \frac{\hbar \omega_0}{2} \cosh \frac{\hbar \omega_0}{k_B T} \cos^2 \gamma_2 t$$

$$+ \frac{\hbar \omega_0}{2} \cosh \frac{\hbar \omega_2}{k_B T} \sin^2 \gamma_2 t. \quad (31)$$

The first term on the right-hand side is the amplified signal, the second term is the amplified noise of the signal circuit $L_1C_1$, and the third term is the down-converted amplified noise from the idler circuit $L_2C_2$.

### 6 Conclusion and Outlook

We have discussed the application of Lagrange and Hamilton methods to quantum circuits consisting of several synchronously DC driven Josephson junctions embedded into linear lossless reciprocal circuits. Further work will consider also dissipation and fluctuation [27] investigations including also unbiased Josephson junctions and Josephson junctions biased at different DC voltages.

### 7 Acknowledgement

The authors thank the German Federal Ministry for Research and Education (BMBF) for financial support of the investigation on Quantum Radar systems within the QUARATE project.

### References


