Analytical Algorithms for Locating Line Sources

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Abstract

Analytical algorithms are investigated for finding the characteristic parameters of a line source radiating inside a homogeneous cylindrical medium. Under certain conditions, the permittivity and permeability of the medium can also be determined. Preliminary results for the effect of boundary data with noise are presented.

1 Introduction

An inverse line-source problem is considered in which we seek to determine the location and current of an electric filament located inside a homogeneous and isotropic cylindrical medium. The utilized measurements are the values of the electric field on the boundary of the medium. The unknown parameters of the problem can be determined analytically by means of suitable normalized Fourier coefficients of the boundary data. In certain cases, the permittivity and permeability of the medium can also be determined analytically. The effect of noisy boundary data on the obtained results is discussed.

Numerical schemes for treating similar two-dimensional inverse problems in electrostatics and magnetostatics were devised in [1]-[4]. Finding an electrostatic or an acoustic point source inside a homogeneous sphere by using appropriate moments on the spherical boundary was investigated in [5].

2 Analytical Solution of the Direct Problem

An infinitely-long circular dielectric cylinder of radius \( a \), with axis along the \( z \)-axis, is excited by an internal \( z \)-directed electric-current filament \( I \) lying on \( (\rho, \phi) = (h, 0) \) of the \( x \)-axis, with \( h < a \). The cylinder has relative dielectric permittivity \( \varepsilon_1 \) and magnetic permeability \( \mu_1 \) and lies in free space with permittivity \( \varepsilon_0 \) and permeability \( \mu_0 \).

The sole \( z \)-component of the primary electric field \( E^p(\rho, \phi) = E^p(\rho, \phi) \hat{z} \) is given by

\[
E^p(\rho, \phi) = AIH_0 \left( k_1 \sqrt{\rho^2 + h^2 - 2\rho h \cos \phi} \right),
\]

where \( H_n \) denote the \( n \)-th order cylindrical Hankel functions of the first-kind \( H_n^{(1)} \), while \( k_0 = \omega \sqrt{\varepsilon_0 \mu_0} \) and \( k_1 = k_0 \sqrt{\varepsilon_1 \mu_1} \) are the external and internal wavenumbers, respectively, and \( A = - (\omega \mu_0 \mu_1) / 4 \).

Imposing the transmission boundary conditions on \( \rho = a \), we obtain the following exact expressions of the \( z \)-components of the secondary and the total electric fields [6]

\[
E^{sc}(\rho, \phi) = AI \sum_{n=-\infty}^{\infty} \beta_n J_n(k_1 \rho) e^{i n \phi}, \quad 0 < \rho < a,
\]

\[
E_0(\rho, \phi) = AI \sum_{n=-\infty}^{\infty} \alpha_n H_n(k_0 \rho) e^{i n \phi}, \quad \rho > a,
\]

where

\[
\alpha_n = - \frac{2i}{a \pi} \frac{J_n(k_1 h)}{k_1 J'_n(k_1 a) H_n(k_0 a) - k_0 \mu_1 H'_n(k_0 a) J_n(k_1 a)},
\]

\[
\beta_n = - J_n(k_1 h) \times \frac{k_1 J'_n(k_1 a) H_n(k_0 a) - k_0 \mu_1 H'_n(k_0 a) H_n(k_1 a)}{k_1 J'_n(k_1 a) H_n(k_0 a) - k_0 \mu_1 H'_n(k_0 a) J_n(k_1 a)},
\]

with \( J_n \) denoting the \( n \)-th order cylindrical Bessel functions.

The total electric field in the interior of the cylinder is expressed, for \( 0 < \rho < a \) with \( (\rho, \phi) \neq (h, 0) \), as

\[
E_i(\rho, \phi) = E^p(\rho, \phi) + E^{sc}(\rho, \phi).
\]

Now, the electric field on the cylinder is given by

\[
E_{cy}(\phi) = E_0(a, \phi) = \hat{\lambda} \mu_1 I \sum_{n=-\infty}^{\infty} \alpha_n H_n(k_0 a) e^{i n \phi},
\]

where \( \hat{\lambda} = - \omega \mu_0 / 4 \). This field will be the basic function that we will use to find the line source and the internal parameters of the cylinder.

3 Inverse Line-Source Problem

In the considered inverse line-source problem, we seek to determine the location \( h \) and the current \( I \) of the line source. Regarding the cylinder’s material parameters \( \varepsilon_1 \) and \( \mu_1 \), we will initially consider them as unknowns and see in which cases they can also be determined analytically.
We define the normalized moments

\[ M_n = \frac{1}{2\pi A H_n(k_0 a)} \int_{-\pi}^{\pi} E_{\text{Cyl}}(\phi) e^{-i\phi} d\phi \]

\[ = \mu_1 I \alpha_n, \quad (8) \]

where the quantities \( A H_n(k_0 a) \) used as normalization coefficients are known for known cylinder's radius \( a \) and free-space parameters.

Suppose that \( k_1 \) is known, but \( \varepsilon_1 \) and \( \mu_1 \) are unknown. We proceed to determine \( h, I, \varepsilon_1 \) and \( \mu_1 \). By considering \( M_n \), we can obtain values for \( \alpha_n \). Since \( k_1 = k_0 \sqrt{\varepsilon_1 \mu_1} \), if we find \( \mu_1 \) then we can also determine \( \varepsilon_1 \) as well.

From the recurrence relations of Bessel functions of integer order, we have

\[ \frac{1}{k_1 h} = \frac{J_{n-1}(k_1 h) + J_{n+1}(k_1 h)}{2 n J_n(k_1 h)} = \frac{M_{n-1} d_{n-1} + M_{n+1} d_{n+1}}{2 n M_n d_n}, \quad (9) \]

where

\[ d_n = k_1 J_n'(k_1 a) H_n(k_0 a) - k_0 \mu_1 H_n'(k_0 a) J_n(k_1 a) \]

\[ = k_1 J_n'(k_1 a) H_n(k_0 a) - k_0 \mu_1 H_n'(k_0 a) J_n(k_1 a) \quad (10) \]

for each \( n \geq 1 \).

Equating two of (9), yields

\[ \frac{M_{n-1} d_{n-1} + M_{n+1} d_{n+1}}{n M_n d_n} = \frac{M_n d_n + M_{n+2} d_{n+2}}{(n + 1) M_{n+1} d_{n+1}}, \quad (11) \]

which gives

\[ (n + 1) M_{n+1} d_{n+1} (M_{n-1} d_{n-1} + M_{n+1} d_{n+1}) = n M_n d_n (M_n d_n + M_{n+2} d_{n+2}). \quad (12) \]

For each \( n \geq 1 \), this is a quadratic equation for \( \mu_1 \) because \( d_n \) is linear in \( \mu_1 \). This equation is written as

\[ A_n \mu_1^2 + B_n \mu_1 + C_n = 0, \quad (13) \]

where

\[ A_n = 2(n + 1) M_n^2 \left[ k_0 H_n'(k_0 a) J_{n+1}(k_1 a) \right]^2 \]

\[ + 2(n + 1) M_{n+1} M_{n-1} \left[ k_0 H_n'(k_0 a) J_{n-1}(k_1 a) \right] \]

\[ H_n'(k_0 a) J_{n+1}(k_1 a) \] - \[ 2 n M_n^2 \left[ k_0 H_n'(k_0 a) J_{n+1}(k_1 a) \right]^2 \]

\[- 2 n M_{n+2} M_{n-2} \left[ k_0 H_n'(k_0 a) J_{n+2}(k_1 a) J_{n+2}(k_1 a) \right] \],

\[ B_n = 2 n M_n^2 \left[ 2 k_0 k_1 J_n'(k_1 a) H_n(k_0 a) H_n'(k_0 a) J_n(k_1 a) \right] \]

\[ + 2 n M_n M_{n+2} + 2 k_0 k_1 \left[ J_n'(k_1 a) H_n(k_0 a) H_n'(k_0 a) J_n(k_1 a) \right] \]

\[- 2(n + 1) M_n^2 \left[ 2 k_0 k_1 J_n'(k_1 a) H_n(k_0 a) \right] \]

\[ H_n'(k_0 a) J_{n+1}(k_1 a) \]

\[- 2(n + 1) M_{n+1} M_{n-1} k_0 k_1 \left[ J_n'(k_1 a) H_n(k_0 a) J_{n+1}(k_1 a) \right] \]

\[ + J_n'(k_1 a) H_n(k_0 a) J_{n+1}(k_1 a), \]

\[ C_n = 2(n + 1) M_{n+1}^2 \left[ k_1 J_n'(k_1 a) H_{n+1}(k_0 a) \right]^2 \]

\[ + 2(n + 1) M_{n+1} M_n k_1^2 J_n'(k_1 a) H_{n+1}(k_0 a) \]

\[ J_n'(k_1 a) H_n(k_0 a) - 2 n M_n^2 \left[ k_1 J_n'(k_1 a) H_n(k_0 a) \right]^2 \]

\[- 2 n M_n M_{n+2} k_1^2 J_n'(k_1 a) H_n(k_0 a) J_{n+2}(k_1 a) H_{n+2}(k_0 a). \]

The permeability \( \mu_1 \) solves (13), for each \( n \geq 1 \). Having determined \( \mu_1 \), the location \( h \) is found using (9), while \( \varepsilon_1 \) follows from the expression of \( k_1 \). The current \( I \) is finally obtained from the moments (8).

### 4 Low-frequency regime

If \( k_1 \) is unknown then an analytical progress can be made in the low-frequency regime. Precisely, for \( \kappa_1 = k_1 a \ll 1 \) and \( \kappa_0 = k_0 a \ll 1 \), the coefficients \( \alpha_n \) have the following leading-order approximations as \( \kappa_1 \to 0 \) and \( \kappa_0 \to 0 \)

\[ \alpha_0 = \frac{1}{\mu_1}, \quad \alpha_n = \frac{(k_0 h)^n}{2^{n-1} n! (1 + \mu_1)}, \quad n \geq 1. \quad (14) \]

Combining the latter with (8), gives

\[ h = \frac{4 \alpha_2}{k_0 \alpha_1} = \frac{4 \alpha_2}{k_0 M_1}. \]

Subsequently, we have \( M_0 = \mu_1 I \alpha_0 = I \) and \( \mu_1 = \frac{M_1}{k_0 \alpha_2} \), and, hence, \( h, I, \) and \( \mu_1 \) are determined.

However, we cannot determine \( k_1 \), and therefore \( \varepsilon_1 \), from the leading-order approximations of \( \alpha_n \), because \( k_1 \) does not appear. To this end, we could use a higher-order approximation, like, e.g.

\[ \alpha_0 \sim \frac{1 - \frac{1}{4} (k_1 h)^2}{(k_1 a)^2 \left( -\frac{1}{4} (k_0 a - 1)^2 + k_0 a - 1 \right) - \mu_1 \left( 1 - \frac{1}{4} (k_1 a)^2 \right)^2}, \quad (15) \]

by which we obtain an estimate for \( k_1 \).
Now, we present the results of some preliminary numerical experiments in the case of noisy data. Particularly, we consider that the measured electric field on $\rho = a$ is given by

$$E_{cyl}^\delta(\phi) = N_{\delta}E_{cyl}(\phi),$$  \hspace{1cm} (16)

where $E_{cyl}(\phi)$ is the true field on $\rho = a$, and $N_{\delta}$ is the noise function, which has the form

$$N_{\delta} = 1 + \delta \text{rand},$$  \hspace{1cm} (17)

where rand gives uniformly distributed random numbers in $[-1,1]$, and $\delta$ is the noise level parameter.

From (8), we see that the noise is inherited from $E_{cyl}$ to the moments $M_n$, namely it holds that the noisy moments are given by

$$M_n^\delta = N_{\delta}M_n.$$  \hspace{1cm} (18)

Moreover, we see that the only unknown parameter that is affected by the noise is $I$, giving the noisy estimate $I^\delta = N_{\delta}I$. Then, for the relative error of the determination of $I$, holds

$$e(\delta) = \frac{|I - I^\delta|}{|I|} \leq \delta.$$  \hspace{1cm} (19)

We add noise to the electric-field data with the noise level $\delta$ ranging from 2% to 20% with a step of 2%. The values of the current $I$ obtained in the presence of the aforementioned noisy data are shown in Table 1, where five iterations for the random noise are depicted (for each value of $\delta$).

Table 1. Determined noisy values $I^\delta$ for different noise levels $\delta$. The true value of the line-source’s current is $I = 4$.

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The average values of the error $e(\delta)$ versus the noise level $\delta$ are depicted in Fig. 1.

References


