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Analytical Algorithms for Locating Line Sources

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Abstract

Analytical algorithms are investigated for finding the characteristic parameters of a line source radiating inside a homogeneous cylindrical medium. Under certain conditions, the permittivity and permeability of the medium can also be determined. Preliminary results for the effect of boundary data with noise are presented.

1 Introduction

An inverse line-source problem is considered in which we seek to determine the location and current of an electric filament located inside a homogeneous and isotropic cylindrical medium. The utilized measurements are the values of the electric field on the boundary of the medium. The unknown parameters of the problem can be determined analytically by means of suitable normalized Fourier coefficients of the boundary data. In certain cases, the permittivity and permeability of the medium can also be determined analytically. The effect of noisy boundary data on the obtained results is discussed.

Numerical schemes for treating similar two-dimensional inverse problems in electrostatics and magnetostatics were devised in [1]-[4]. Finding an electrostatic or an acoustic point source inside a homogeneous sphere by using appropriate moments on the spherical boundary was investigated in [5].

2 Analytical Solution of the Direct Problem

An infinitely-long circular dielectric cylinder of radius a, with axis along the z-axis, is excited by an internal z-directed electric-current filament I lying on $(\rho,\phi)=(h,0)$ of the x-axis, with $h<a$. The cylinder has relative dielectric permittivity ε_1 and magnetic permeability μ_1 and lies in free space with permittivity ε_0 and permeability μ_0.

The sole z-component of the primary electric field $\mathbf{E}^{\mathrm{pr}}(\rho,\phi)=E^{\mathrm{pr}}(\rho,\phi)\hat{\mathbf{z}}$ is given by

$$E^{\mathrm{pr}}(\rho,\phi)=AIH_0\left(k_1\sqrt{\rho^2+h^2-2\rho h\cos\phi}\right), \qquad (1)$$

where H_n denote the n-th order cylindrical Hankel functions of the first-kind $H_n^{(1)}$, while $k_0=\omega\sqrt{\varepsilon_0\mu_0}$ and $k_1=k_0\sqrt{\varepsilon_1\mu_1}$ are the external and internal wavenumbers, respectively, and $A=-(\omega\mu_0\mu_1)/4$.

Imposing the transmission boundary conditions on $\rho=a$, we obtain the following exact expressions of the z-components of the secondary and the total electric fields [6]

$$E^{\mathrm{sec}}(\rho,\phi)=AI\sum_{n=-\infty}^{\infty}\beta_n J_n(k_1\rho)e^{\mathrm{i}n\phi}, \quad 0<\rho<a, \qquad (2)$$

$$E_0(\rho,\phi)=AI\sum_{n=-\infty}^{\infty}\alpha_n H_n(k_0\rho)e^{\mathrm{i}n\phi}, \quad \rho>a, \qquad (3)$$

where

$$\alpha_n=-\frac{2\mathrm{i}}{a\pi}\times \frac{J_n(k_1h)}{k_1J_n'(k_1a)H_n(k_0a)-k_0\mu_1H_n'(k_0a)J_n(k_1a)}, \qquad (4)$$

$$\beta_n=-J_n(k_1h)\times \frac{k_1H_n'(k_1a)H_n(k_0a)-k_0\mu_1H_n'(k_0a)H_n(k_1a)}{k_1J_n'(k_1a)H_n(k_0a)-k_0\mu_1H_n'(k_0a)J_n(k_1a)}, \qquad (5)$$

with J_n denoting the n-th order cylindrical Bessel functions.

The total electric field in the interior of the cylinder is expressed, for $0<\rho<a$ with $(\rho,\phi)\neq(h,0)$, as

$$E_1(\rho,\phi)=E^{\mathrm{pr}}(\rho,\phi)+E^{\mathrm{sec}}(\rho,\phi). \qquad (6)$$

Now, the electric field on the cylinder is given by

$$E_{\mathrm{cyl}}(\phi)=E_0(a,\phi)=\tilde{A}\mu_1I\sum_{n=-\infty}^{\infty}\alpha_nH_n(k_0a)e^{\mathrm{i}n\phi}, \qquad (7)$$

where $\tilde{A}=-\omega\mu_0/4$. This field will be the basic function that we will use to find the line source and the internal parameters of the cylinder.

3 Inverse Line-Source Problem

In the considered inverse line-source problem, we seek to determine the location h and the current I of the line source. Regarding the cylinder's material parameters ε_1 and μ_1, we will initially consider them as unknowns and see in which cases they can also be determined analytically.

We define the normalized moments

$$M_n = \frac{1}{2\pi \tilde{A} H_n(k_0 a)} \int_{-\pi}^{\pi} E_{\text{cyl}}(\phi) e^{-\mathrm{i}n\phi} \mathrm{d}\phi = \mu_1 I \alpha_n, \tag{8}$$

where the quantities $\tilde{A}H_n(k_0 a)$ used as normalization coefficients are known for known cylinder's radius a and free-space parameters.

Suppose that k_1 is known, but ε_1 and μ_1 are unknown. We proceed to determine h, I, ε_1 and μ_1. By considering M_n, we can obtain values for α_n. Since $k_1 = k_0\sqrt{\varepsilon_1 \mu_1}$, if we find μ_1 then we can also determine ε_1 as well.

From the recurrence relations of Bessel functions of integer order, we have

$$\frac{1}{k_1 h} = \frac{J_{n-1}(k_1 h) + J_{n+1}(k_1 h)}{2n\, J_n(k_1 h)} = \frac{M_{n-1} d_{n-1} + M_{n+1} d_{n+1}}{2n\, M_n d_n}, \tag{9}$$

where

$$d_n = k_1 J_n'(k_1 a) H_n(k_0 a) - k_0 \mu_1 H_n'(k_0 a) J_n(k_1 a) \tag{10}$$

for each $n \geq 1$.

Equating two of (9), yields

$$\frac{M_{n-1} d_{n-1} + M_{n+1} d_{n+1}}{n\, M_n d_n} = \frac{M_n d_n + M_{n+2} d_{n+2}}{(n+1) M_{n+1} d_{n+1}}, \tag{11}$$

which gives

$$(n+1) M_{n+1} d_{n+1} (M_{n-1} d_{n-1} + M_{n+1} d_{n+1}) = nM_n d_n (M_n d_n + M_{n+2} d_{n+2}). \tag{12}$$

For each $n \geq 1$, this is a quadratic equation for μ_1 because d_n is linear in μ_1. This equation is written as

$$A_n \mu_1^2 + B_n \mu_1 + C_n = 0, \tag{13}$$

where

$$\begin{aligned} A_n =& 2(n+1) M_{n+1}^2 \Big[k_0 H_{n+1}'(k_0 a) J_{n+1}(k_1 a)\Big]^2 \\ &+ 2(n+1) M_{n+1} M_{n-1} \Big[k_0^2 H_{n-1}'(k_0 a) J_{n-1}(k_1 a) \\ &H_{n+1}'(k_0 a) J_{n+1}(k_1 a)\Big] - 2nM_n^2 \Big[k_0 H_n'(k_0 a) J_n(k_1 a)\Big]^2 \\ &- 2nM_n M_{n+2} \Big[k_0^2 H_n'(k_0 a) J_n(k_1 a) H_{n+2}'(k_0 a) J_{n+2}(k_1 a)\Big], \end{aligned}$$

$$\begin{aligned} B_n =& 2nM_n^2 \Big[2k_0 k_1 J_n'(k_1 a) H_n(k_0 a) H_n'(k_0 a) J_n(k_1 a)\Big] \\ &+ 2nM_n M_{n+2} k_0 k_1 \Big[J_n'(k_1 a) H_n(k_0 a) H_{n+2}'(k_0 a) \\ &J_{n+2}(k_1 a) + J_{n+2}'(k_1 a) H_{n+2}(k_0 a) H_n'(k_0 a) J_n(k_1 a)\Big] \\ &- 2(n+1) M_{n+1}^2 \Big[2k_0 k_1 J_{n+1}'(k_1 a) H_{n+1}(k_0 a) \\ &H_{n+1}'(k_0 a) J_{n+1}(k_1 a)\Big] - 2(n+1) M_{n+1} M_{n-1} k_0 k_1 \\ &\Big[J_{n-1}'(k_1 a) H_{n-1}(k_0 a) H_{n+1}'(k_0 a) J_{n+1}(k_1 a) \\ &+ J_{n+1}'(k_1 a) H_{n+1}(k_0 a) H_{n-1}'(k_0 a) J_{n-1}(k_1 a)\Big], \end{aligned}$$

$$\begin{aligned} C_n =& 2(n+1) M_{n+1}^2 \Big[k_1 J_{n+1}'(k_1 a) H_{n+1}(k_0 a)\Big]^2 \\ &+ 2(n+1) M_{n+1} M_{n-1} k_1^2 J_{n-1}'(k_1 a) H_{n-1}(k_0 a) \\ &J_{n+1}'(k_1 a) H_{n+1}(k_0 a) - 2nM_n^2 \Big[k_1 J_n'(k_1 a) H_n(k_0 a)\Big]^2 \\ &- 2nM_n M_{n+2} k_1^2 J_n'(k_1 a) H_n(k_0 a) J_{n+2}'(k_1 a) H_{n+2}(k_0 a). \end{aligned}$$

The permeability μ_1 solves (13), for each $n \geq 1$. Having determined μ_1, the location h is found using (9), while ε_1 follows from the expression of k_1. The current I is finally obtained from the moments (8).

4 Low-frequency regime

If k_1 is unknown then analytical progress can be made in the low-frequency regime. Precisely, for $\kappa_1 = k_1 a \ll 1$ and $\kappa_0 = k_0 a \ll 1$, the coefficients α_n have the following leading-order approximations as $\kappa_1 \to 0$ and $\kappa_0 \to 0$

$$\alpha_0 = \frac{1}{\mu_1}, \quad \alpha_n = \frac{(k_0 h)^n}{2^{n-1} n! (1+\mu_1)}, \; n \geq 1. \tag{14}$$

Combining the latter with (8), gives

$$h = \frac{4\alpha_2}{k_0 \alpha_1} = \frac{4M_2}{k_0 M_1}.$$

Subsequently, we have $M_0 = \mu_1 I \alpha_0 = I$ and $\mu_1 = \frac{M_1}{Ik_0 h - M_1}$, and, hence, h, I, and μ_1 are determined.

However, we cannot determine k_1, and therefore ε_1, from the leading-order approximations of α_n, because k_1 does not appear. To this end, we could use a higher-order approximation, like, e.g.

$$\alpha_0 \sim \frac{1 - \frac{1}{4}(k_1 h)^2}{(k_1 a)^2 \left(-\frac{1}{2}(k_0 a - 1)^2 + k_0 a - 1\right) - \mu_1 \left(1 - \frac{1}{4}(k_1 a)^2\right)}, \tag{15}$$

by which we obtain an estimate for k_1.

5 Numerical Results

Now, we present the results of some preliminary numerical experiments in the case of noisy data. Particularly, we consider that the measured electric field on $\rho = a$ is given by

$$E_{\rm cyl}^{\delta}(\phi) = N_{\delta}\, E_{\rm cyl}(\phi), \tag{16}$$

where $E_{\rm cyl}(\phi)$ is the true field on $\rho = a$, and N_{δ} is the noise function, which has the form

$$N_{\delta} = 1 + \delta\, rand, \tag{17}$$

where *rand* gives uniformly distributed random numbers in $[-1, 1]$, and δ is the noise level parameter.

From (8), we see that the noise is inherited from $E_{\rm cyl}$ to the moments M_n, namely it holds that the noisy moments are given by

$$M_n^{\delta} = N_{\delta}\, M_n. \tag{18}$$

Moreover, we see that the only unknown parameter that is affected by the noise is I, giving the noisy estimate $I^{\delta} = N_{\delta} I$. Then, for the relative error of the determination of I, holds

$$e(\delta) \equiv \frac{|I - I^{\delta}|}{|I|} \leq \delta. \tag{19}$$

We add noise to the electric-field data with the noise level δ ranging from 2% to 20% with a step of 2%. The values of the current I obtained in the presence of the aforementioned noisy data are shown in Table 1, where five iterations for the random noise are depicted (for each value of δ).

Table 1. Determined noisy values I^{δ} for different noise levels δ. The true value of the line-source's current is $I = 4$.

δ% \ iter #	1	2	3	4	5
2	3.9359	3.9271	4.0092	4.0436	3.9699
4	4.0413	3.8725	3.9651	3.8575	4.0004
6	3.8260	3.9472	4.2051	4.2004	4.1025
8	4.0937	4.2132	3.9349	4.1599	4.2145
10	4.0956	3.8885	4.2052	3.9311	3.9939
12	4.4360	3.5506	3.8626	4.1561	3.7902
14	3.4933	3.8306	3.9455	3.7098	4.2409
16	3.5374	4.1129	3.8287	4.3927	4.0048
18	3.3409	4.6811	3.5525	4.2407	4.1245
20	3.3342	4.7597	4.2422	3.5700	3.8456

The average values of the error $e(\delta)$ versus the noise level δ are depicted in Fig. 1.

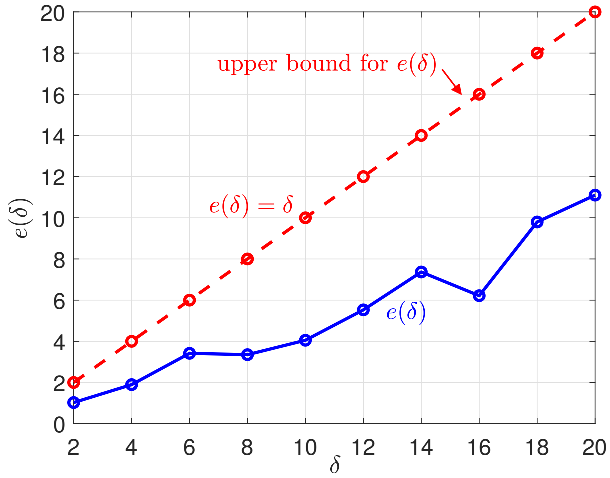

Figure 1. Average values of the relative errors $e(\delta)$ versus the noise level δ for ten iterations for the random noise. The line $e(\delta) = \delta$ specifying the upper bound of the relative errors is also depicted. The true value of the line-source's current is $I = 4$.

References

[1] L. Baratchart, A. Ben Abda, F. Ben Hassen, and J. Leblond, "Recovery of pointwise sources or small inclusions in 2D domains and rational approximation," *Inverse Problems*, **21**, 2005, pp. 51–74, doi.org/10.1088/0266-5611/21/1/005.

[2] A. Ben Abda, F. Ben Hassen, J. Leblond, and M. Mahjoub, "Sources recovery from boundary data: a model related to electroencephalography," *Math. Comput. Modelling*, **49**, 2009, pp. 2213–2223, doi.org/10.1016/j.mcm.2008.07.016.

[3] D. Kandaswamy, T. Blu, and D. Van De Ville D, "Analytic sensing: noniterative retrieval of point sources from boundary measurements," *SIAM J. Sci. Comput.*, **31**, 2009, pp. 3179–3194, doi.org/10.1137/080712829.

[4] M. Hanke and W. Rundell, "On rational approximation methods for inverse source problems," *Inverse Problems Imaging*, **5**, 2011, pp. 185–202, doi: 10.3934/ipi.2011.5.185.

[5] N. L. Tsitsas and P. A. Martin, "Finding a source inside a sphere," *Inverse Problems*, **28**, 2012, 015003 (11pp), doi:10.1088/0266-5611/28/1/015003.

[6] C. A. Valagiannopoulos, N. L. Tsitsas, and G. Fikioris, "Convergence analysis and oscillations in the method of fictitious sources applied to dielectric scattering problems," *J. Optical Soc. America A*, **29**, 2012, pp. 1–10, doi.org/10.1364/JOSAA.29.000001.