An Application of Monte Carlo Markov Chains in Inverse Electromagnetic Scattering Problems

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Abstract

To tackle the nonlinearity in inverse electromagnetic scattering problems, we present a novel application of Markov Chain Monte Carlo (MCMC) methods to infer permittivity values. Instead of using regularizations to uncover the best fit of permittivity, we estimate the conditional mean of the unknown permittivity given scattered field data. The conditional mean estimates not only incorporate prior knowledge from results obtained by Born Iterative Methods (BIM), but also avoid the nonlinearity by computing the linear forward model. For a homogeneous cylinder with the relative permittivity of 11, numerical results of BIM are improved by MCMC.

1 Introduction

Inverse electromagnetic scattering problems arise in many scientific and engineering fields [1, 2, 3]. Given a scatterer immersed in free space and an incident field, the electromagnetic scattering phenomenon is characterized by the electric field integral equation as

\[
E'(r) = -k_0^2 \int_V (\epsilon(r') - 1)G(r,r') \cdot E(r') \, dV', \quad r \notin V, \quad (1)
\]

where \(E'(r)\) is the scattered electric field, \(k_0\) is the wave number in free space, \(\epsilon(r')\) is the complex permittivity profile of the scatterer, \(G(r,r')\) is the Green’s function, and \(E(r')\) is the total field within the scatterer. The time harmonic \(e^{j\omega t}\) is omitted throughout this paper.

For a forward problem, one aims to compute the scattered field with a known permittivity profile. As \(E(r')\) is numerically determined from the known \(\epsilon(r')\), \(E'(r)\) can be solved with a fixed integrand. For an inverse problem, given the scattered electric field, we aim to reconstruct the complex permittivity profile. As \(E(r')\) becomes an unknown due to the unknown \(\epsilon(r')\), the inverse problem for the permittivity is nonlinear.

Some previous deterministic approaches linearize the nonlinear inverse scattering problem. For example, the first-order Born approximation corresponds to the initial step of Born-related methods such as the Born iterative method (BIM) [4] and the distorted Born iterative method [5]. Similarly, higher-order approximations do not linearize our problem, yet provide more accurate representations of unknown fields [6], which result in better reconstructions.

Some previous stochastic approaches combine Born approximation and natural computing techniques, such as genetic algorithms [7] and memetic algorithms [8]. Also Bayesian inference has been widely used in parameter estimation, for example, in electrical impedance tomography [9].

In this work, we extend the work presented in [10]. First, we adapt BIM results as priors by treating the unknown permittivity of a two-dimensional scatterer and known scattered field data as random variables. Then we sample the conditional mean (CM) integration with a simple MCMC method, using the Metropolis-Hastings algorithm to construct the chain by random walks with Gaussian proposal functions. Numerical computations based on the Method of Moment (MoM) for a homogeneous lossless scatterer with a known contour are performed.

2 Theory

To numerically solve two dimensional inverse scattering problems, one discretizes Eq. 1 into matrix form with the Method of Moment [11]. Let there be \(N_{tx}\) transmitters and \(N_{rx}\) receivers around the scatterer under investigation; there are \(N_{tx} \times N_{rx}\) measurements. Given the tomographic configuration, the scatterer is illuminated by one transmitter each time, and the corresponding scattered fields are measured by all receivers. Therefore, multi-illumination and multi-static measurements are achieved to infer information about this scatterer. Then the domain of the scatterer is discretized into \(n\) pixels. At the \(m\)-th measurement, the scattered field \(E^s\) can be rewritten as

\[
E^s_m = -\frac{jk_0^2}{4} \sum_{i=1}^{n} (\epsilon_i - 1) E_i \int_{\mathcal{S}} H_0^{(2)}(k_0|\mathbf{r}_m - \mathbf{r}_i|) \, d\mathcal{S}', \quad (2)
\]

where \(H_0^{(2)}\) is the Hankel function of the second kind, \(\mathbf{r}_m \notin \mathcal{S}\). By enforcing the “measurement” location to be within the scatterer, the total field at the \(p\)-th pixel, \(E_p\), can be represented as

\[
E_p = E^i_p - \frac{jk_0^2}{4} \sum_{i=1}^{n} (\epsilon_i - 1) E_i \int_{\mathcal{S}} H_0^{(2)}(k_0|\mathbf{r}_p - \mathbf{r}_i|) \, d\mathcal{S}', \quad (3)
\]

where \(E^i_p\) is the incident field at \(p\)-th pixel, \(\mathbf{r}_p \in \mathcal{S}\).

The BIM routine starts with the Born approximation in Eq. 2 to compute the permittivity profile, which then updates the field within the scatterer by Eq. 3. In this work,
we use the conjugate gradient (CG) method as the regulariza-
tion technique to solve the inversion in Eq. 2 due to
the ill-conditioned property of the linear operator. After
the BIM method provides stable outputs, we reformulate
this inverse problem from a Bayesian inference perspective
and consider measurements and parameters of a statistical
model as random variables.

2.1 Bayesian Inference and MCMC

The Bayes theorem states that the distribution of unknown
parameters $X$ conditioned on scattered field data $D$ is

$$P(X|D) = \frac{P(X)P(D|X)}{P(D)}.$$  (4)

In Bayesian inverse models, the solution of an inverse prob-
lem takes the form of a posterior probability distribution,
$P(X|D)$, which is proportional to the prior multiplied by
the likelihood. The likelihood, $P(D|X)$, of data $D$ given per-
mittivity parameters $X$ is strongly associated with the forward
scattering model. Plausible priors of $X$ are: 1) the real part
of the relative permittivity being more than or equal to 1
and the imaginary part being negative; 2) previous recon-
struction results from deterministic conjugate gradient regu-
larizations offering upper and lower bounds.

To estimate the permittivity parameters, we choose the con-
tditional mean, i.e. the center of the posterior probability
distribution, of the unknown model parameter $X$,

$$X_{CM} = E\{X|D\} = \int_{x} X\pi(X|D)\,dX,$$  (5)

where $\pi(X|D)$ is the posterior density. The conditional
mean estimate solves an integration problem, so usually it
is more robust towards noises in the data than the maximum
a posteriori estimate.

Since the high dimension of the unknown discrete param-
eters requires a large sample space, it is challenging to nu-
merically evaluate the integration for a conditional mean
estimate. Here, the conditional mean is sampled in a sta-
tistical sense using MCMC methods, which can be applied
favorably for our nonlinear inverse problem as they only
 depend on the forward model. The Monte Carlo integration
draws samples from the posterior probability density and
takes the average of these samples. Thus, we approximate
the integral in Eq. 5 with the population mean

$$E\{X|D\} \approx \frac{1}{N} \sum_{i=1}^{T} X_i \pi(X_i|D).$$  (6)

To sufficiently draw samples from posterior distributions,
we construct a Markov chain. With the Metropolis-
Hastings (MH) algorithm, a Markov chain converges to its
stationary distribution, which is also the posterior distribu-
tion we are trying to sample. At each time $t$, one samples $Y$
from the proposal distribution. According to the acceptance
t ratio, the sample $Y$ is either accepted as the next state $X_{t+1}$
or not. In this work, we use the simplest MH algorithm, the
random walk with a multivariate Gaussian proposal func-
tion, which has the proposal distribution,

$$q(Y|X) = q(|Y - X|).$$  (7)

This is a trial-and-error strategy; at each state $t$, we add
some randomness to $X_t$ so that the proposed sample $Y$ ex-
plorcs the solution space.

3 Numerical Results and Analysis

For a simple test case, we choose an infinitely long circular
cylinder with the radius of $\lambda/20$ and the relative permitt-
itivity of 11. This object is borrowed from [4]. A square
that contains the circle with the side length of 0.03 m is the
investigation domain. To avoid the inverse crime, the inves-
tigation domain, $S$, is divided into finer pixels ($144 \times 144$)
in the forward model than the inverse one ($36 \times 36$). There
are 8 transmitters and 36 receivers that offer 288 measure-
ments. $TM$-mode incident fields are radiated by a line
source at 1 GHz and scattered fields are numerically calcu-
lated by MoM codes.

First, given the contour of the scatterer, we perform the tra-
ditional BIM with the conjugate gradient; Figure 1 shows
the BIM results after 11 iterations: a rough range for the
real part of the permittivity and quite accurate reconstruc-
tions of the conductivity.

Then, for the real part of permittivity, we set the max/min
values acquired in BIM as the upper/lower bounds for the
permittivity random variable; for the imaginary part, we
assume it is 0. The starting point of the Markov chain is
the mean of the real permittivity at all pixels in the scat-
ter. As previous BIM results offer a good starting point,
we don’t throw away any iterations at the beginning of the
chain; therefore no burn-in phase is need. At each iteration,
a random variable of the normal distribution with a standard
development of 0.02 is added to the current state of permittiv-
ity values; this procedure generates a sample permittivity,
which would be sent to the forward model to obtain the
sample data. The difference between these sample data and
scattered field data determines if this sample permittivity
would be accepted as the next state of the chain. The longer
the chain is, the closer the estimate is to the true posterior.

Figure 2 shows the reconstructed results of MCMC after
10000 iterations. The acceptance rate is 0.233. Due to the
random nature of MCMC, the reconstructed permi-
tivity profile in Figure 2(a) is not as smooth as the con-
jugate gradient results in Figure 1(a). However, even if the
lower/upper bounds are set as [9.5, 13], the condi-
tional mean estimate by MCMC offers narrower bounds as
[9.766, 12.815]. Moreover, the mean of the permittivity at
all pixels by BIM is 11.180, which is improved by MCMC
as 11.022; the standard deviation by BIM is 1.043, which
is also improved by MCMC as 0.669. Figure 2(b) directly
Figure 1. BIM reconstructed results for the permittivity within a cylinder, $\varepsilon = 11$: (a) real part; (b) conductivity.

Figure 2. Reconstructed results for real part of the permittivity: (a) MCMC results within the cylinder; (b) slice comparisons along the horizontal axis.
compares the permittivity reconstructions along 36 horizontal pixels at $y = 0$ by BIM with the conjugate gradient and MCMC.

Furthermore, we insert those reconstructed permittivity values into the forward model to compute scattered fields, which are compared with the analytical scattered field shown as in Figure 3. Not surprisingly, for forward model results, MCMC improves both of the amplitude and the phase of scattered fields. Since scattered fields are complex, we compare the amplitude by calculating the error as

$$\text{error} = \left| \frac{E_{\text{CG,MCMC}} - E_{\text{data}}}{E_{\text{data}}} \right|.$$  \hfill (8)

![Figure 3. Comparison of scattered fields from the forward model. Left axis: error percentage of amplitude. Right axis: phase.](image)

4 Conclusions

Stochastic Bayesian inference is applied with priors from the BIM with the conjugate gradient technique. MCMC is computationally expensive compared to deterministic regularizations; however, MCMC does improve the reconstructed permittivity profile for a scatterer with $\varepsilon = 11$, where the Born approximation might fail. Future work would include applying MCMC to inhomogeneous scatterers and adding noise to data for robustness tests.

References


