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Dual Spaces for 3D Finite Element Methods

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The finite element method for the Maxwell cavity problem provides approximate solutions to the time-harmonic vector wave equation

$$\operatorname{curl}^2 e - \kappa^2 e = f, \quad \text{in } \Omega, \tag{1}$$

with κ the wave number, subject to the perfectly conducting boundary condition $e \times n = 0$, on $\Gamma = \partial\Omega$ [4]. It does so by considering a variational formulation for (1): find e that is curl-conforming and fulfills the boundary condition, such that for all e', also curl-conforming and fulfulling the boundary condition, it holds that

$$\int_\Omega \left(\operatorname{curl} e' \cdot \operatorname{curl} e - \kappa^2 e' \cdot e\right) = \int_\Omega e' \cdot f \tag{2}$$

Approximate solutions are constructed by restricting e and e' to a finite dimensional subspace of the space of candidate solutions. The magnetic field can be computed by either (i) application of Faraday's law, or (ii) moving to a first order formulation of the cavity problem [2, 3]. Both the electric field and magnetic field are (in lowest order versions of the finite element method), approximated by expansions in curl-conforming Nédélec edge elements (g_i). This precludes the option to compute the magnetic field as $h = -\frac{1}{i\mu\omega}\operatorname{curl} e$, which results in an expansion in div-conforming elements.

In this contribution, we will present a finite element space $\operatorname{span}(\tilde{f}_k)$ that is dual to the space of curl-conforming Nédélec edge-elemements. The space will be constructed on a realisation of the Voronoi dual in the barycentric refinement of a primal tetrahedral mesh [1]. The functions are constructed by first fixing their divergence on the cells of the dual mesh, and then choosing the remaining degrees of freedom to maximise their approximation accuracy. They can be used to compute accurate and conforming approximations of the magnetic field by enforcing the weak equality

$$\sum_{j\in\mathscr{E}'} \left(\int_\Omega \tilde{f}_k \cdot g_j\right) y_j = -\frac{1}{\imath\mu\omega} \sum_{i\in\mathscr{E}} \left(\int_\Omega \tilde{f}_k \cdot \operatorname{curl} g_i\right) x_i, \quad \forall k \in \mathscr{E}', \tag{3}$$

with $\mathscr{E}'$ the set of all edges in the mesh, and $\mathscr{E}$ the set of all edges not on the boundary. To solve this system for the expansion coefficients (y_j) for the magnetic fields, the matrix in the left hand side needs to be inverted. Note that this matrix is square and contains only $\mathscr{O}(|\mathscr{E}|)$ non-zero entries. Numerical results will demonstrate that it is well-conditioned allowing for efficient computation of the magnetic field.

References

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