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Direct Solver Based on the Multilevel Nonuniform Grid Approach for Quasi-Planar Electromagnetic Scattering Problems

Evgeny V. Chernokozhin* (1) and Amir Boag(1)
(1) Tel Aviv University, Tel Aviv 69978, Israel

Abstract

A direct solver for quasi-planar electromagnetic scattering problems is developed. The algorithm consists of a multilevel domain decomposition, a rank revealing procedure (RRP), and a direct solution of the resulting compressed system of linear algebraic equations. The RRP is performed step by step from the bottom to top levels and is based on the singular value decomposition (SVD) using the interpolation grids of the Multilevel Nonuniform Grid (MLNG) approach. The efficiency of the solver is demonstrated by numerical experiments.

1 Introduction

Electromagnetic (EM) scattering problems are often solved using the Method of Moments (MoM). This method has important advantages of accuracy and generality, in the sense that it can be applied to scatterers of arbitrary shape. However, it also has serious limitations in frequency and scatterer sizes, due to its high computational complexity. Indeed, in the MoM, the integral equation describing the EM scattering problem is reduced to a system of linear algebraic equations with a fully populated matrix. The direct solution by the Gauss method or its modifications requires $O(N^3)$ operations, where N is the number of unknowns in the system. Iterative methods have a computational complexity (CC) of $O(N^2)$ per iteration, but this value should be multiplied by the number of iterations, which, in poorly conditioned/resonant cases, can be fairly large.

A significant advantage of the direct methods is the possibility to solve the problems with multiple excitations, i.e., with many right-hand sides (RHSs) simultaneously, and also in the cases of poor convergence—this makes the direct methods so attractive. Many efforts have been made to develop direct methods for scattering problems. Most of them involve a rank-revealing procedure in one or another form, aimed at the matrix compression. The approach is based on the observation that different parts of the scatterer interact with each other via much fewer degrees of freedom than that involved in self-interactions within these parts. This fact is reflected in the rank deficiency of the operators representing the interactions between different parts of the scatterer. The rank deficiency can be significant when special formulations of the scattering problem are used [1] or under some restrictions on the geometry of the scatterer, e.g., for strongly elongated [2] or planar [3] scatterers. As a result, the initial system of linear algebraic equations can be reduced to solving a linear system of significantly lower dimensions.

The method presented in this work was designed for the case of quasi-planar scatterers and has been implemented in a fast numerical solver. It combines the rank revealing procedure with the MLNG approach, earlier developed for iterative solvers of acoustic and EM scattering problems [4, 5]. The well-tested MLNG approach is used here instead of skeletonization, traditionally used in direct solvers.

2 Statement of the Problem

The scatterer S is assumed to be a set of perfectly conducting infinitely thin open surfaces in free space, lying within a planar layer of thickness substantially smaller than the size of S and the wavelength of the incident field. The EM scattering problem is formulated in the form of an electric field integral equation (EFIE)

$$i\omega \mathbf{A}_\tau(\mathbf{r}) + \nabla_\tau \Phi(\mathbf{r}) = \mathbf{E}_\tau^{\text{inc}}(\mathbf{r}), \quad \mathbf{r} \in S \tag{1}$$

with τ indicating the tangential to S field component. The RHS in (1), $\mathbf{g} = \mathbf{E}_\tau^{\text{inc}}$, represents the excitation, while $\mathbf{A}$ and Φ are the vector and scalar potentials:

$$\mathbf{A}(\mathbf{r}) = \mu \int_S G(\mathbf{r}-\mathbf{r}')\mathbf{J}(\mathbf{r}')ds', \tag{2}$$

$$\Phi(\mathbf{r}) = \frac{i}{\omega\varepsilon} \int_S G(\mathbf{r}-\mathbf{r}')\nabla\cdot\mathbf{J}(\mathbf{r}')ds', \tag{3}$$

where $G(\mathbf{r}-\mathbf{r}')$ is the scalar free-space Green's function.

To discretize integral equation (1), S is triangulated, a set of N RWG basis functions (BFs) is introduced, and the current $\mathbf{J}$ is sought in the form of a linear combination of BFs. The testing with N testing functions reduces the problem to a system of N linear algebraic equations.

3 The MLNG Algorithm

The fast direct solver presented here essentially employs the MLNG algorithm [4]. Originally, this algorithm was

developed for the efficient calculation of the integrals of the type

$$\Psi(\mathbf{r}) = \int_S G(\mathbf{r}-\mathbf{r}')\varphi(\mathbf{r}')ds', \qquad (4)$$

with a scalar density $\varphi(\mathbf{r}')$. Representation (4) is equally applicable to the Cartesian components of the vector potential (2) and to scalar potential (3), and the MLNG algorithms was modified for solving problems formulated based on the EFIE. Here, we give its brief description including the details essential for the direct solver (for details see [4]).

The domain of integration S is divided into a hierarchy of progressively smaller subdomains $S_n^{(l)}$ of progressively lower levels, where $l = 0,1,\ldots,L$ is the level and n is the subdomain's index within the l-th level. At the top level, $S_1^{(0)} = S$. Parent–child relations between subdomains are established. Since S is quasi-planar, each domain can have up to four child subdomains, which results in the formation of a generally incomplete quadtree. For each subdomain, except for $l = 0$ and 1, the near and interpolation zones are defined, and to each of them a spherical interpolation grid $\Gamma_n^{(l)}$ is assigned. Each grid $\Gamma_n^{(l)}$ allows interpolation to the observation points on S within the interpolation zone of $S_n^{(l)}$ and to the grids $\Gamma_m^{(l-1)}$ assigned to the parent domains $S_m^{(l-1)}$. For each observation point $\mathbf{r} \in S$, we denote by $S_2(\mathbf{r})$ the union of 2nd-level subdomains to whose near zones $\mathbf{r}$ does not belong; by $S_l(\mathbf{r})$, the union of all lth level subdomains from $S\backslash(S_2(\mathbf{r})\cup\ldots\cup S_{l-1}(\mathbf{r}))$ to whose near zones $\mathbf{r}$ does not belong; and, by $\mathrm{N}(\mathbf{r})$, we denote the complement $S\backslash(S_2(\mathbf{r})\cup\ldots\cup S_L(\mathbf{r}))$. Thus, for each observation point $\mathbf{r} \in S$, we have a unique decomposition of the domain of integration S into nonintersecting sets:

$$S = \mathrm{N}(\mathbf{r})\cup S_2(\mathbf{r})\cup\ldots\cup S_l(\mathbf{r})\cup\ldots\cup S_L(\mathbf{r}).$$

The field produced by sources φ on S at point $\mathbf{r} \in S$ can be represented by the sum of integrals over these sets. The integral over $\mathrm{N}(\mathbf{r})$ is calculated directly. The integrals over the sets $S_l(\mathbf{r})$, which consist of selected subdomains $S_n^{(l)}$, are evaluated by the interpolation to $\mathbf{r}$ of the phase- and amplitude-compensated potentials $\mathbf{A}$ and Φ stored on the grids $\Gamma_n^{(l)}$ assigned to them. The result of interpolation is multiplied by the phase- and amplitude-restoration factor, which depends on the point $\mathbf{r}$ and the subdomain $S_n^{(l)}$. The role of this factor and its reciprocal, the phase- and amplitude-compensation factor, is to mitigate the rapidly oscillating behavior of the field and improve the accuracy of interpolation. The fields on the grids $\Gamma_n^{(L)}$, assigned to bottom-level subdomains, are obtained by the direct integration. The potentials $\mathbf{A}$ and Φ on all other grids $\Gamma_n^{(l)}$ are obtained by the interpolation from the grids $\Gamma_m^{(l+1)}$ assigned to the child subdomains. The procedure described above enables one to calculate the surface field for a given source field φ by asymptotically $O(N\log N)$ rather than $O(N^2)$ operations needed when the same testing conditions are satisfied using the impedance matrix.

4 Partial Operators

Hereinafter, we use the same designations for both integral operations and their discretized approximations. We represent Eq. (1) in the form $\mathrm{Z}\mathbf{J} = \mathbf{g}$ and define partial operators $\mathrm{Z}_n^{(l)}$ as the operator Z in which the integration is restricted to the domain $S_n^{(l)}$ and the observation points also lie in $S_n^{(l)}$. We obviously have $\mathrm{Z}_1^{(0)} = \mathrm{Z}$. We also define operators

$$\mathrm{Z}^{(l)} = \sum_{n=1}^{N_l} \mathrm{Z}_n^{(l)}, \qquad (5)$$

which consist of independent "diagonal" blocks acting within nonintersecting domains of the same level. Then, operator Z can be represented in the form

$$\mathrm{Z} = \mathrm{Z}^{(L)} + \sum_{l=0}^{L-1}\left(\mathrm{Z}^{(l)} - \mathrm{Z}^{(l+1)}\right). \qquad (6)$$

We also define operators $\mathrm{K}^{(l)} = \mathrm{Z}^{(l)} - \mathrm{Z}^{(l+1)}$, consisting of operators $\mathrm{Z}_n^{(l)}$ from which the diagonal blocks of the (l+1)-st level have been removed. Thus, the integral equation to be solved can be represented as

$$\mathrm{Z}^{(L)}\mathbf{J} + \sum_{l=0}^{L-1}\mathrm{K}^{(l)}\mathbf{J} = \mathbf{g}. \qquad (7)$$

We assume that operator $Z^{(L)}$ is invertible. Defining operators $\mathrm{X}_{L-1,l} = \left[\mathrm{Z}^{(L)}\right]^{-1}\mathrm{K}^{(l)}$ and, successively, for $k = 2,\ldots,L-1$,

$$\mathrm{X}_{L-k,l} = \left(I + \mathrm{X}_{L-k+1,L-k+1}\right)^{-1}\ldots\left(I + \mathrm{X}_{L-1,L-1}\right)^{-1}\mathrm{X}_{L,l}$$

we find the formal solution of Eq. (7) in factorized form:

$$\mathbf{J} = \left(I + \mathrm{X}_{1,1}\right)^{-1}\ldots\left(I + \mathrm{X}_{L-1,L-1}\right)^{-1}\left[\mathrm{Z}^{(L)}\right]^{-1}\mathbf{g}. \qquad (8)$$

We assume that each operators $\mathrm{X}_{k,k}$ can be approximately represented in a factorized form $\mathrm{X}_{k,k} \approx UV^*$ (e.g., by means of truncated singular value

decomposition (SVD)) with $N\times r$ matrices U and V, where $r<N$ and V is formed by orthogonal unit vectors. Then, each operator $I+\mathrm{X}_{k,k}$ can be inverted as a sum of an invertible and a finite-dimensional operators. The solution of Eq. (7) is obtained by successively solving relatively simple equations with rank deficient operators.

5 Rank Revealing Procedure

First, we factorize the bottom-level operators $\bar{Z}_n^{(L)}=Z-Z_n^{(L)}$, i.e., operators of the interaction of a subdomain $S_n^{(L)}$ with its complement to S. To reduce the CC of this procedure, instead of $\bar{Z}_n^{(L)}$, we consider its substitute $\tilde{Z}_n^{(L)}$, which has the same interaction with the near zone of $S_n^{(L)}$ as $\bar{Z}_n^{(L)}$, but the interaction with the interpolation zone of $S_n^{(L)}$ is replaced by the interaction with the interpolation grid $\Gamma_n^{(L)}$. Instead of finding the SVD of $\bar{Z}_n^{(L)}$, we find the SVD of $\tilde{Z}_n^{(L)}$, $\tilde{Z}_n^{(L)}\approx\tilde{U}_n^{(L)}\tilde{D}_n^{(L)}\tilde{V}_n^{(L),*}$. Since the values of the potentials at the nodes of $\Gamma_n^{(L)}$ completely determine the field in the interpolation zone of $S_n^{(L)}$, we have a formal factorization $\bar{Z}_n^{(L)}\approx\left(\bar{Z}_n^{(L)}\tilde{V}_n^{(L)}\right)\tilde{V}_n^{(L),*}$.

Then, in ascending order of levels, we factorize all operators $\bar{Z}_n^{(l)}=Z-Z_n^{(l)}$. Like for the bottom level, instead of $\bar{Z}_n^{(l)}$, we consider their substitutes $\tilde{Z}_n^{(l)}$, defined as follows. For $\bar{Z}_n^{(l)}$, we already have the factorizations of all operators $\bar{Z}_n^{(l+1)}$, of its child domains and, in particular, the corresponding matrices $\tilde{V}_n^{(l+1)}$. Operator $\tilde{Z}_n^{(l)}$ is the mapping of all basis vectors from the matrices $\tilde{V}_n^{(l+1)}$ to the interpolation grid $\Gamma_n^{(l)}$, to the interpolation grids of all relevant (i.e., participating in the interpolation to the near zone) descendant subdomains, and to the near zones of all relevant bottom-level subdomains. Finding the SVD $\tilde{Z}_n^{(l)}\approx\tilde{U}_n^{(l)}\tilde{D}_n^{(l)}\tilde{V}_n^{(l),*}$ gives us the matrix $\tilde{V}_n^{(l)}$, from which we obtain a formal factorization $\bar{Z}_n^{(l)}\approx\left(\bar{Z}_n^{(l)}\tilde{V}_n^{(l)}\right)\tilde{V}_n^{(l),*}$.

After factorizing $\bar{Z}_n^{(l)}$, in a similar manner, using the corresponding matrices $\tilde{V}_n^{(l)}$, we obtain the factorizations of the operators $\mathrm{K}^{(l)}=\mathrm{Z}^{(l)}-\mathrm{Z}^{(l+1)}$.

6 Required Storage and Computational Complexity

In the case of direct solvers, the required storage and CC as functions of N are determined by the number of degrees of freedom in the system, which essentially depends on the geometry of the scatterer. For planar scatterers, the ranks of the operators $\bar{Z}_n^{(l)}$ are approximately proportional to the perimeter of the domains $S_n^{(l)}$ rather than to their area. In this case, the storage required for the direct solver under consideration is theoretically estimated by $O(N^{3/2})$ and the CC by $O(N^{3/2}\log^2 N)$. For strongly elongated scatters, these values are substantially smaller. Obviously, these estimates do not hold for non-planar scatterers, but may be considered theoretical limits achievable for problems with scatterers slightly deviating from planar.

7 Numerical Experiments

The algorithm was implemented in a FORTRAN code. The benchmark measurements were performed on a series of problems with scatterers consisting of progressively increasing number of infinitely thin PEC patches, with N varying in the range ~10^3–10^6. To control the accuracy, we used a "conventional" MoM iterative solver and, beyond its range, the MLNG iterative solver [5]. The accuracy and computational time essentially depend on the threshold for the singular values. In this series of experiments, the accuracy was maintained within $5\cdot10^{-3}$ in the norms of L_1, L_2, and L_∞. For planar scatterers, the exponent α in the dependence $O(N^\alpha)$ of the computation time proved to be about 1.5.

8 References

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