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Filtering via Graph Laplacians: a Hierarchical-Basis-Free Multilevel Decomposition for the Electric Field Integral Equation

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Abstract

The solution of the electric field integral equation at low frequency and/or high discretization regimes is one of main computational bottleneck that compromises the efficiency and accuracy of this equation. Therefore, effective preconditioner is a must in order to reconstruct the scattered field in many problems of practical interest. In this paper we present a new preconditioner that efficiently stabilizes the EFIE. It is based on multilevel spectral filters built from primal and dual graph Laplacians. The sparsity of the latter ensures an accelerated convergence with a minimal computational overhead.

1 Introduction

The Electric Field Integral Equation (EFIE) has become a widely used computational tool for simulating electromagnetic radiation and PEC (Perfect Electric Conductor) scattering. The key attribute of this scheme, in comparison with differential equation-based method, is the relatively small matrices it yields to, as it only requires the discretization of the boundary of the scatterer. While the system matrix obtained via the BEM (Boundary Element Method) discretization of the EFIE is dense, its iterative solution can be reduced to quasi-linear complexity by using acceleration techniques such as the Multi-Level Fast Multipole Method (MLFMM) [1]. The optimum performance of the latter, however, is impeded by conditioning issues of the EFIE occurring at low frequency and high-discretization regimes [2]. The low frequency breakdown is due to the inverse scaling of the two operators of the EFIE, whereas the high discretization breakdown is due to the EFIE's diverging spectral branches [3]. The combination of those two downsides prevents the use of standard EFIE to problems of practical applications, because an ill-conditioned system results not only in an increase in the number of iterations used by an iterative solver, but also in a loss of accuracy. Therefore, a preconditioner that cures both breakdowns is indispensable for the modelization of electromagnetic fields at low frequency and in the presence of complex geometries.

In order to overcome both breakdowns, various alternative have been devised, prominent among which are Calderón identity-based techniques. This class of preconditioners exploits the analytic properties of the EFIE operator, which states that squaring the EFIE operator results in an identity plus a compact operator [4]; hence a linear system with bounded condition number. Standard implementation of this regularizer, however, requires the use of the barycentric refinement of the original mesh, thus increasing the computational overhead. Another class of preconditioners that do not require the use of dual basis functions are Hierarchical basis preconditioners [5]. Operating in the Sobolev norm induced by the EFIE, those preconditioners yield a well-conditioned system that is independent of h-refinement. Unfortunately however, those class of preconditioners necessitate structured meshes and often results in a non-multiplicative preconditioner. The first drawback drastically limits their application and the second complicate their incorporation in existing EFIE codes.

In this work, we present a new operator-based preconditioner which avoids the use of barycentric-refined mesh by directly operating on the original one. Its multiplicative nature and quasi-linear complexity facilitate its integration with existing codes while maintaining the optimum performance of standard acceleration techniques. The new preconditioner leverages on graph Laplacian operators to turn the EFIE into a well-conditioned integral equation. Its straightforward implementation, however, would require the expensive computation of fractional power and inverses of the Laplacian, thus jeopardizing the efficiency of the regularizer. In order to overcome this limitation, we present a set of strategies that operate in wavelet-like fashion to efficiently build the preconditioner. In particular, this preconditioner regularizes the two spectral branches of the EFIE block by block with multilevel filters built from primal and dual graph Laplacian matrices. Given the sparsity of the latter, the EFIE is regularized at low computational effort. Several numerical results validate the presented scheme in canonical and realistic geometries. Preliminary results about this scheme were presented in [6]

2 Background and notations

Consider a perfect electric conducting object Ω embedded in a space characterized by a permittivity ε and permeability μ. Let Γ be a Lipschitz surface representing the boundary of Ω and $\hat{\boldsymbol{n}}$ its outward pointing unit normal. A time harmonic electromagnetic wave $\boldsymbol{E}^i$ impinges on Ω inducing an electric current on its surface, which then radiates the scattered field $\boldsymbol{E}^s$. The latter can be computed by solving the EFIE which reads

$$-\hat{\boldsymbol{n}}(\boldsymbol{r}) \times \boldsymbol{E}^{\boldsymbol{i}} = \mathcal{T}\boldsymbol{J} = \mathcal{T}_A\boldsymbol{J} + \mathcal{T}_\phi\boldsymbol{J} \tag{1}$$

where $\mathcal{T}_A\boldsymbol{J}$ and $\mathcal{T}_\phi\boldsymbol{J}$, being the vector potential and the scalar potential respectively, are defined as

$$\mathcal{T}_A\boldsymbol{J} = \hat{\boldsymbol{n}}(\boldsymbol{r}) \times \mathrm{j}k \int_\Gamma \frac{\mathrm{e}^{\mathrm{j}k\|\boldsymbol{r}-\boldsymbol{r}'\|}}{4\pi\|\boldsymbol{r}-\boldsymbol{r}'\|}\boldsymbol{J}(\boldsymbol{r}')\mathrm{d}S(\boldsymbol{r}') \tag{2}$$

$$\mathcal{T}_\phi\boldsymbol{J} = -\hat{\boldsymbol{n}}(\boldsymbol{r}) \times \frac{1}{\mathrm{j}k}\nabla_{\boldsymbol{r}} \int_\Gamma \frac{\mathrm{e}^{\mathrm{j}k\|\boldsymbol{r}-\boldsymbol{r}'\|}}{4\pi\|\boldsymbol{r}-\boldsymbol{r}'\|}\nabla_{\boldsymbol{r}'}\cdot\boldsymbol{J}(\boldsymbol{r}')\mathrm{d}S(\boldsymbol{r}') \tag{3}$$

where $k = \omega\sqrt{\varepsilon\mu}$ denotes the wavenumber. This equation is then numerically solved by (i) discretizing Γ into a triangular mesh of average edge length h; (ii) expanding the current density $\boldsymbol{J}$ as a linear combination of div-conforming (such as RWG) basis functions $\boldsymbol{J} = \sum_{n=1}^{N} I_n \boldsymbol{f}_n(\boldsymbol{r})$ [7]; (iii) testing the discretized EFIE with rotated RWG basis functions $\hat{\boldsymbol{n}}(\boldsymbol{r}) \times \boldsymbol{f}_n(\boldsymbol{r})$ to obtain a linear system $\boldsymbol{T}\boldsymbol{j} = \boldsymbol{e}$ where the left hand side is

$$\boldsymbol{T} = \mathrm{j}k\boldsymbol{T}_A + (\mathrm{j}k)^{-1}\boldsymbol{T}_\phi \tag{4}$$

in which

$$\boldsymbol{T}_A = \langle \hat{\boldsymbol{n}}(\boldsymbol{r}) \times \boldsymbol{f}_n(\boldsymbol{r}), \mathcal{T}_A(\boldsymbol{f}_n(\boldsymbol{r}))\rangle \tag{5}$$

$$\boldsymbol{T}_\phi = \left\langle \hat{\boldsymbol{n}}(\boldsymbol{r}) \times \boldsymbol{f}_n(\boldsymbol{r}), \mathcal{T}_\phi(\boldsymbol{f}_n(\boldsymbol{r}))\right\rangle \tag{6}$$

and the right hand side is $\boldsymbol{e} = \left\langle \boldsymbol{f}_n(\boldsymbol{r}), -\boldsymbol{E}^i\right\rangle$. With a condition number that grows as $\mathrm{cond}(\boldsymbol{T}) \lesssim 1/(hk)^2$, solving the linear system of equation (4) iteratively becomes inconceivable at low frequency and dense discretization regimes.

To cope with the low-frequency breakdown, the loop-star decomposition has proved successful [8, 9]. This quasi-Helmholtz decomposition separates the solenoidal and non-solenoidal components of the current $\boldsymbol{J}$ for a subsequent suitable rescaling in frequency. In particular, let $\boldsymbol{\Lambda}$ and $\boldsymbol{\Sigma}$ be the transformation matrices that map the RWG space into the loop and star subspaces respectively. Then, a left and right preconditioner of the form $\boldsymbol{\Upsilon} = [\boldsymbol{\Lambda}/\sqrt{k} \quad \boldsymbol{\Sigma}\sqrt{k}]$ eliminates the frequency related ill-conditioning. While with this technique, the EFIE can be accurately solved at the lower end of the spectrum, it does not however cure the h-refinement breakdown. In fact, the condition number of $\boldsymbol{T}_{LS} = \boldsymbol{\Upsilon}^{\mathrm{T}}\boldsymbol{T}\boldsymbol{\Upsilon}$ deteriorates as $\mathrm{cond}(\boldsymbol{T}_{LS}) \lesssim 1/h^3$.

3 The multilevel preconditioner

Preconditioning the h quadratic growth of the $\boldsymbol{T}_{LS}$ system requires an intrusive spectral regularization of two components of the EFIE. Specifically, in the limit $k \to 0$, the discretization of the vector potential with loop functions is equivalent to the static hypersingular operator discretized with pyramid basis functions $\langle \lambda, \mathcal{N}(\lambda)\rangle$ [10]. It is well known that this operator is unbounded having a derivative strength of order 1 [11]. Using left and right preconditioning with the graph Laplacian of fractional order, the $\mathcal{N}$ operator is amenable to a well-conditioned equation. Similarly, in the limit $k \to 0$, the discretization of the scalar potential is related to the static single layer operator discretized with patch basis functions $\langle \Pi, \mathcal{S}(\Pi)\rangle$. This compact operator of order -1 can also be regularized with the dual graph Laplacian matrix of fractional order. Computing fractional Laplacian, however requires the computationally intensive eigendecomposition and results in a dense matrix, hence making it unsuitable for many problems of practical interests. In order to avoid this drawback, we propose a properly tailored multi-level spectral filters to regularize the EFIE. This is efficiently achieved through sparse operations involving only the forward Laplacian matrices. In particular, let $\boldsymbol{P}_i^{\Lambda}$ and $\boldsymbol{P}_i^{\Sigma}$ define low-pass filters

$$\boldsymbol{P}_i^{\Sigma} = \frac{\mathbf{I}}{\mathbf{I} + \left(\frac{\boldsymbol{\Delta}^{\Sigma}}{2^i}\right)^n} \quad \text{for } i = 1 \ldots \log_2(\sigma_{\max}^{\Sigma}), \tag{7}$$

$$\boldsymbol{P}_i^{\Lambda} = \frac{\mathbf{I}}{\mathbf{I} + \left(\frac{\boldsymbol{\Delta}^{\Lambda}}{2^i}\right)^n} \quad \text{for } i = 1 \ldots \log_2(\sigma_{\max}^{\Lambda}), \tag{8}$$

where $\boldsymbol{\Delta}^{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma}^{\mathrm{T}}\boldsymbol{\Sigma}$ and $\boldsymbol{\Delta}^{\boldsymbol{\Lambda}} = \boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\Lambda}$ are the the primal and dual graph Laplacians respectively. Subsequently, we build the preconditioner as $\boldsymbol{Q}_i = (\boldsymbol{P}_i - \boldsymbol{P}_{i-1})\frac{1}{\sqrt{\sigma_{\max}^i}}$ in which the normalization constant $\sigma_{\max}^i$ is the largest singular-value of the spectral interval $\left[2^{i-1}, 2^i\right]$. Finally we get the regularized EFIE

$$\begin{bmatrix} \boldsymbol{Q}_i^{\Sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{Q}_i^{\Lambda} \end{bmatrix} \begin{bmatrix} k\widetilde{\boldsymbol{\Sigma}}^{\mathrm{T}}(\boldsymbol{T}_{\boldsymbol{A}} + \boldsymbol{T}_{\boldsymbol{\phi}})\widetilde{\boldsymbol{\Sigma}} & \widetilde{\boldsymbol{\Sigma}}^{\mathrm{T}}\boldsymbol{T}_{\boldsymbol{A}}\boldsymbol{\Lambda} \\ \boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{T}_{\boldsymbol{A}}\widetilde{\boldsymbol{\Sigma}} & \frac{1}{k}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{T}_{\boldsymbol{A}}\boldsymbol{\Lambda} \end{bmatrix} \begin{bmatrix} \boldsymbol{Q}_i^{\Sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{Q}_i^{\Lambda} \end{bmatrix} \tag{9}$$

where $\widetilde{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}^{\mathrm{T}}\boldsymbol{\Sigma}\right)^{+}$.

4 Numerical Results

As a first example, we consider a unit sphere discretized into 3000 triangles and illuminated with a plane wave oscillating at different frequency. Correspondingly, a study of the condition number of the standard EFIE is reported in figure 1 showing the low frequency breakdown. Using our new technique, the condition number stays stably small till arbitrary low frequency; thus making the EFIE completely immune from the low frequency breakdown.

In the second numerical test, we keep the frequency of the plane wave fixed 1 Hz, while increasing the discretization density. Figure 2 reports the condition number of the EFIE matrix regularized with loop-star preconditioner. It is evident that the corresponding condition number deteriorates as the average edge length decreases. Upon preconditioning with our technique, the EFIE is regularized against dense discretization breakdown.

As a final numerical experiment, we illuminated an aircraft, discretized into 19k facets, with a 10^5 Hz plane wave. Preconditioning the EFIE with the Laplacian spectral filters,

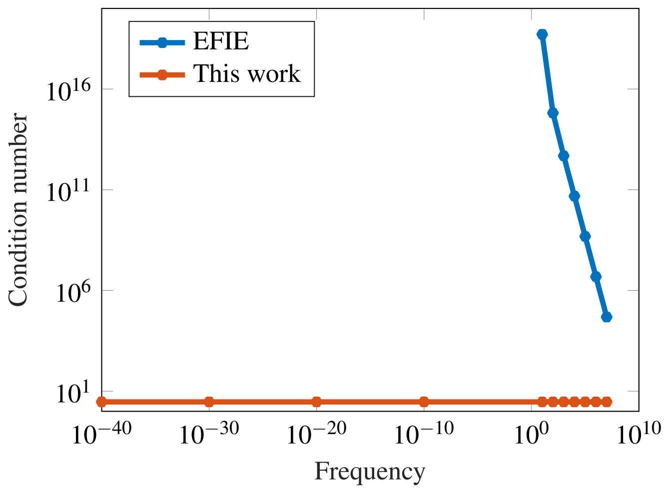

Figure 1. Condition number as a function of the frequency

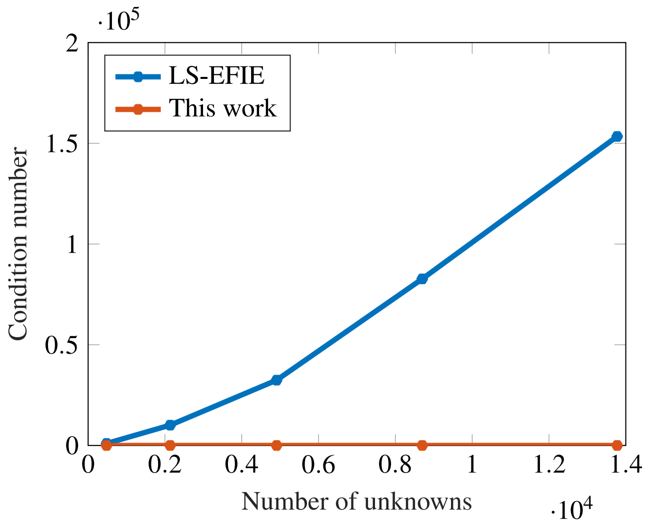

Figure 2. Condition number as a function of number of elements

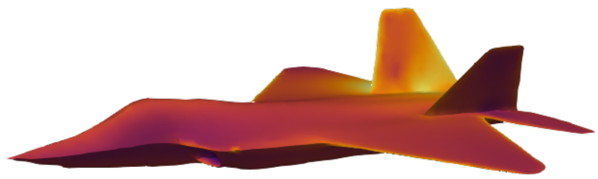

Figure 3. The induced surface current

the iterative solver converged to the current solution, shown on 3, in 500 iterations.

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