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Noise and Dissipation in a Circuit Quantum Electrodynamic Description of a Josephson Traveling-Wave Parametric Amplifier

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Abstract

In this paper, we are constructing a circuit quantum electrodynamic model describing dissipative effects in a Josephson traveling-wave parametric amplifier (JTWPA). Losses and noise are treated using the quantum Langevin method. The total Hamiltonian of our model describes the coupling of the modes in a parametric four-wave-mixing scheme, as well as the interaction with the environment. Based on the Heisenberg equations of motion, we obtain the temporal evolution of the signal photon number and analyze the signal gain of the JTWPA.

1 Introduction

Superconducting nanoelectronic devices operating in the microwave frequency range have found applications in quantum-state engineering and quantum computing [1–4]. In recent years, superconducting quantum circuits have been developed based on the Josephson effect [5], i.e. the tunneling of superconducting electron pairs through a thin insulating barrier between two superconductors.

One particularly interesting application of the Josephson effect lies in the utilization of the nonlinear Josephson inductance for parametric amplification of microwave signals with very low power levels. A Josephson parametric amplifier (JPA) was proposed at first in [6] and general energy rules for Josephson junctions (JJs) governing the operation of JJ oscillators, detectors, mixers, and parametric amplifiers were formulated in [7]. It has been shown in [8, 9] that the noise in a JPA approaches the quantum limit. Therefore, JPAs are commonly used as the first amplifier stage for the readout of transmon qubit states [10]. A flux-driven JPA including a coplanar-waveguide resonator was presented in [11]. A flux-modulated JPA based on an array of superconducting quantum interference devices (SQUIDs) has been shown to exhibit a high gain of up to 31 dB [12]. A first quantum mechanical description of the JPA was given in [13], where a circuit representation of the amplifier was used to derive a quantum mechanical model based on canonical quantization.

The operation of JPAs is based on the power exchange between resonant circuits or resonator modes at the signal, idler and pump frequencies. This limits the bandwidth of the JPA. This limitation can be overcome by using a traveling-wave type architecture as proposed by [14–16]. In [15] and [16] the nonlinear transmission line structure featuring discrete Josephson elements is described by a nonlinear wave equation using classical circuit theory. Quantum mechanical descriptions of JTWPAs in terms of non-dissipative systems are given in [17] and [18]. In this work, we will start from the quantum description as given in [18] and extend the model to treat dissipation and noise by coupling the microwave modes to a Langevin heat bath.

In section 2, we will introduce the amplification mechanism of a JTWPA in the four-wave-mixing case and discuss the phase-matching condition. A quantum theoretical model including noise and dissipation is established by coupling the system with a heat bath in thermal equilibrium in section 3. The Heisenberg equations of motion for the pump, signal, and idler modes are derived under the assumptions of a strong classical pump and a memoryless, Markovian environment. In section 4, analytic solutions for the system of inhomogeneous ordinary differential equations are presented and we discuss the temporal evolution of the photon number in the signal mode and the gain for the case of number states as initial conditions.

2 Traveling Wave Parametric Amplifiers based on Josephson Junctions

An array of capacitively shunted Josephson junctions, embedded in a transmission line similar to [18], is considered. In a typical configuration, such a nonlinear transmission line consists of hundreds or thousands of unit cells, as depicted in Fig. 1. Due to the third-order nonlinearity of the Josephson junctions, parametric amplification by a four-wave-mixing process between a small input signal and a strong pump tone occurs. Energy is transferred between the pump wave at ω_p, the signal wave at ω_s, and the idler wave at ω_i by mixing of pump and signal waves. In the case of degenerate pumping, the frequency-relation is given by $2\omega_\mathrm{p} = \omega_\mathrm{s} + \omega_\mathrm{i}$. Hence, by four-wave-mixing, two pump photons are scattered into a pair of signal and idler photons.

Due to self-phase-modulation (SPM) and cross-phase-modulation (XPM) originating from the strong pump cur-

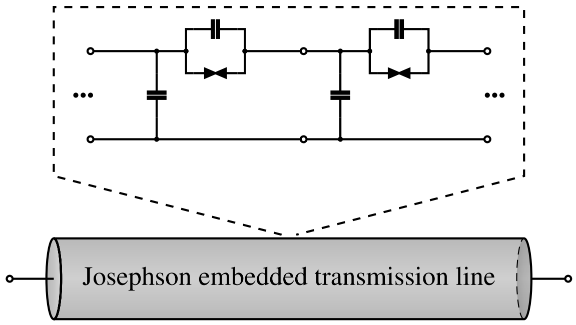

Figure 1. Circuit diagram of a JTWPA.

rent, a phase mismatch Δk_{T} between different modes occurs. The total phase mismatch contains a linear part Δk_{L} and a nonlinear part Δk_{NL} which reduces amplification. The phase mismatch can be compensated by resonant phase matching networks according to Fig. 2 along the transmission line [16, 17, 19].

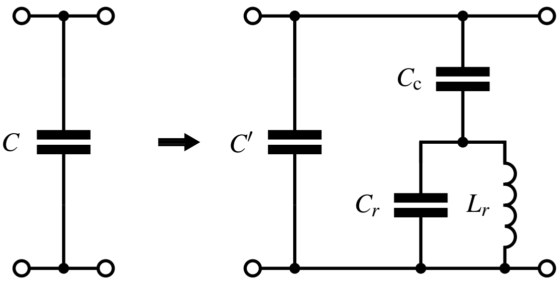

Figure 2. Impedance of a JTWPA with phase shifter (adapted from [17]).

3 System Description and Noise Modeling

We assume four-wave-mixing in a nondegenerate amplifier with a degenerate pump. We consider unidirectionally propagating waves and apply the approximate Hamiltonian $\hat{H}^{\mathrm{CP}}_{\mathrm{TWPA}}$ according to [18] for the non-dissipative system by assuming a strong classical pump. If pump depletion can be neglected, the equation of motion for the pump wave A_{p} can be directly obtained from $\hat{H}_{\mathrm{TWPA}}$ such that

$$\partial_t A_{\mathrm{p}} = \mathrm{i}\omega'_{\mathrm{p}} A_{\mathrm{p}}\,, \qquad \Rightarrow \qquad A_{\mathrm{p}} = A_{\mathrm{p0}} \exp\left(-\mathrm{i}\omega'_{\mathrm{p}} t\right), \quad (1)$$

with $\omega'_{\mathrm{p}} = \omega_{\mathrm{p}} + 2\xi'_{\mathrm{p}} \left|A_{\mathrm{p}}\right|^2$ where we can set $\left|A_{\mathrm{p}}\right|^2 = \left|A_{\mathrm{p0}}\right|^2$. The constant ξ'_{p} is related to the parameters of the Josephson junction and the dispersion relation given in [18].

In order to treat noise and dissipation in the quantum mechanical JTWPA model, we apply the noise operator method from reservoir theory [20, 21]. Analogous to the classical Langevin approach, the heat bath is represented by a large number of coupled harmonic oscillators with densely spaced frequencies ω_k and associated heat bath operators $\hat{b}_k$. The total Hamiltonian of the heat bath-coupled system is given by

$$\hat{H}_{\mathrm{Total}} = \hat{H}_{\mathrm{System}} + \hat{H}_{\mathrm{Bath}} + \hat{H}_{\mathrm{Coupling}}\,. \quad (2)$$

Taking the approximated Hamiltonian for the case of a classical pump $\hat{H}^{\mathrm{CP}}_{\mathrm{TWPA}}$ from [18], the total Hamiltonian is given by

$$\begin{aligned}\hat{H}^{\mathrm{CP}}_{\mathrm{Total}} &= \hat{H}^{\mathrm{CP}}_{\mathrm{TWPA}} + \hat{H}_{\mathrm{Bath}} + \hat{H}_{\mathrm{Coupling}} \\ &= \hbar\left(\omega_{\mathrm{s}} + \xi'_{\mathrm{s}}\left|A_{\mathrm{p}}\right|^2\right)\hat{a}^\dagger_{\mathrm{s}}\hat{a}_{\mathrm{s}} + \hbar\left(\omega_{\mathrm{i}} + \xi'_{\mathrm{i}}\left|A_{\mathrm{p}}\right|^2\right)\hat{a}^\dagger_{\mathrm{i}}\hat{a}_{\mathrm{i}} \\ &\quad - \hbar\left(\chi' A_{\mathrm{p}}^2 \hat{a}^\dagger_{\mathrm{s}}\hat{a}^\dagger_{\mathrm{i}} + \mathrm{H.c.}\right) + \hbar\sum_k \omega_k\left(\hat{b}^\dagger_k\hat{b}_k + \frac{1}{2}\right) \\ &\quad + \hbar\sum_k g_{\mathrm{s}k}(\omega_k)\left(\hat{a}_{\mathrm{s}}\hat{b}^\dagger_k + \hat{b}_k\hat{a}^\dagger_{\mathrm{s}}\right) \\ &\quad + \hbar\sum_k g_{\mathrm{i}k}(\omega_k)\left(\hat{a}_{\mathrm{i}}\hat{b}^\dagger_k + \hat{b}_k\hat{a}^\dagger_{\mathrm{i}}\right). \end{aligned} \quad (3)$$

where ξ'_{s} and ξ'_{i} are constants related to the Josephson parameters and to the dispersion relation, and χ' is a factor related to the wave vector of the pump wave as detailed in [18]. The coupling strength of the signal and idler modes to the heat bath is given by the coupling constants $g_{\mathrm{s}k}(\omega_k)$ and $g_{\mathrm{i}k}(\omega_k)$.

From the Hamiltonian (3), the Heisenberg equations of motion for the annihilation operators of the signal and idler modes and the k-th reservoir mode are found in a straightforward manner to be

$$\partial_t \hat{a}_{\mathrm{s}} = -\mathrm{i}\omega'_{\mathrm{s}}\hat{a}_{\mathrm{s}} + \mathrm{i}\chi' A_{\mathrm{p}}^2 \hat{a}^\dagger_{\mathrm{i}} - \mathrm{i}\sum_k g_{\mathrm{s}k}(\omega_k)\hat{b}_k\,, \quad (4)$$

$$\partial_t \hat{a}_{\mathrm{i}} = -\mathrm{i}\omega'_{\mathrm{i}}\hat{a}_{\mathrm{i}} + \mathrm{i}\chi' A_{\mathrm{p}}^2 \hat{a}^\dagger_{\mathrm{s}} - \mathrm{i}\sum_k g_{\mathrm{i}k}(\omega_k)\hat{b}_k\,, \quad (5)$$

$$\partial_t \hat{b}_k = -\mathrm{i}\omega_k\hat{b}_k - \mathrm{i}g_{\mathrm{s}k}(\omega_k)\hat{a}_{\mathrm{s}} - \mathrm{i}g_{\mathrm{i}k}(\omega_k)\hat{a}_{\mathrm{i}}\,, \quad (6)$$

with $\omega'_j = \omega_j + \xi'_j\left|A_{\mathrm{p0}}\right|^2$, where $j \in \{\mathrm{s},\mathrm{i}\}$.

4 Quantum Mechanical Solutions for the Dynamic System

To find an analytic solution for the equations of motion of the dissipative quantum system (4)-(6), the bath operators $\hat{b}_k$ are eliminated by formal integration of (6). Next, we introduce a rotating frame by $\hat{A}_j = \hat{a}_j \exp\left(\mathrm{i}\omega'_j t\right)$ and then $\tilde{\hat{A}}_j = \hat{A}_j \exp\left(\mathrm{i}/2\left[2\omega'_{\mathrm{p}} - \omega'_{\mathrm{s}} - \omega'_{\mathrm{i}}\right]t\right)$ with $j \in \{\mathrm{s},\mathrm{i}\}$, where non-resonant terms are dropped.

The equations of motion for the annihilation operator of the signal mode and the creation operator of the idler mode in the co-rotating frame can be written in matrix form as

$$\begin{bmatrix}\partial_t\tilde{\hat{A}}_{\mathrm{s}} \\ \partial_t\tilde{\hat{A}}^\dagger_{\mathrm{i}}\end{bmatrix} = \begin{bmatrix}-\gamma_{\mathrm{ss}} + \frac{\mathrm{i}\Delta\Omega}{2} & \mathrm{i}\chi' A_{\mathrm{p0}}^2 \\ -\mathrm{i}\chi'^*\left(A_{\mathrm{p0}}^2\right)^* & -\gamma_{\mathrm{ii}} - \frac{\mathrm{i}\Delta\Omega}{2}\end{bmatrix}\begin{bmatrix}\tilde{\hat{A}}_{\mathrm{s}} \\ \tilde{\hat{A}}^\dagger_{\mathrm{i}}\end{bmatrix} + \begin{bmatrix}\tilde{\hat{f}}_{\mathrm{s}} \\ \tilde{\hat{f}}^\dagger_{\mathrm{i}}\end{bmatrix}, \quad (7)$$

with the Langevin noise operators for the j-th mode

$$\tilde{\hat{f}}_j = -\mathrm{i}\sum_k g_{jk}\hat{b}_{k0}\exp\left[-\mathrm{i}\left(\omega_k - \omega'_j - \frac{\Delta\Omega}{2}\right)t\right], \quad (8)$$

where $j \in \{\mathrm{s}, \mathrm{i}\}$ and the phase mismatch $\Delta\Omega = 2\omega_\mathrm{p}' - \omega_\mathrm{s}' - \omega_\mathrm{i}'$. The damping constants $\gamma_\mathrm{ss}, \gamma_\mathrm{ii} \in \mathbb{R}$ corresponding to the inverse photon lifetime in the respective modes [21] are given by

$$\gamma_\mathrm{ss} = \pi\mathscr{D}\left(\omega_\mathrm{s}' + \frac{\Delta\Omega}{2}\right) g_\mathrm{sk}^2\,, \qquad \gamma_\mathrm{ii} = \pi\mathscr{D}\left(\omega_\mathrm{i}' + \frac{\Delta\Omega}{2}\right) g_\mathrm{ik}^2\,, \tag{9}$$

where $\mathscr{D}(\omega)$ is the density of states in the heat bath. The eigenvalues of the matrix in (7) are given by

$$\lambda_\pm = -\frac{\gamma_\mathrm{ss} + \gamma_\mathrm{ii}}{2} \pm g\,, \tag{10}$$

where g is the gain factor

$$g = \sqrt{\frac{(-\mathrm{i}\Delta\Omega + \gamma_\mathrm{ss} - \gamma_\mathrm{ii})^2}{4} + |\chi'|^2 \left|A_\mathrm{p0}^2\right|^2}\,. \tag{11}$$

An approximate analytical solution of (7) for the signal mode annihilation operator $\tilde{\hat{A}}_\mathrm{s}$ is given by

$$\begin{aligned}
\tilde{\hat{A}}_\mathrm{s} = &\left[\left(\cosh(gt) + \frac{-\gamma_\mathrm{ss} + \gamma_\mathrm{ii} + \mathrm{i}\Delta\Omega}{2g}\sinh(gt)\right)\tilde{\hat{A}}_\mathrm{s0}\right.\\
&\left. + \frac{\mathrm{i}\chi' A_\mathrm{p0}^2}{g}\sinh(gt)\tilde{\hat{A}}_\mathrm{i0}^\dagger\right]\mathrm{e}^{-\frac{\gamma_\mathrm{ss}+\gamma_\mathrm{ii}}{2}t}\\
&+ \sum_k g_{\mathrm{s}k}\hat{b}_{k0}\left[\eta_-(\omega_k) - \eta_+(\omega_k)\right]\mathrm{e}^{-\mathrm{i}\left(\omega_k - \omega_\mathrm{s}' - \frac{\Delta\Omega}{2}\right)t}\\
&+ \sum_k g_{\mathrm{s}k}\hat{b}_{k0}\left[\eta_+(\omega_k)\mathrm{e}^{-gt} - \eta_-(\omega_k)\mathrm{e}^{gt}\right]\mathrm{e}^{-\frac{\gamma_\mathrm{ss}+\gamma_\mathrm{ii}}{2}t}\\
&+ \sum_k g_{\mathrm{i}k}^*\hat{b}_{k0}^\dagger\left[\rho_+(\omega_k) - \rho_-(\omega_k)\right]\mathrm{e}^{\mathrm{i}\left(\omega_k - \omega_\mathrm{i}' - \frac{\Delta\Omega}{2}\right)t}\\
&+ \sum_k g_{\mathrm{i}k}^*\hat{b}_{k0}^\dagger\left[\rho_-(\omega_k)\mathrm{e}^{gt} - \rho_+(\omega_k)\mathrm{e}^{-gt}\right]\mathrm{e}^{-\frac{\gamma_\mathrm{ss}+\gamma_\mathrm{ii}}{2}t}\,,
\end{aligned} \tag{12}$$

where we have for $\eta_\pm(\omega)$ and $\rho_\pm(\omega)$

$$\eta_\pm(\omega) = \frac{\mathrm{i}}{g}\frac{\gamma_\mathrm{ss} - \gamma_\mathrm{ii} - \mathrm{i}\Delta\Omega \pm 2g}{\pm 4g - 4\mathrm{i}\left(\omega - \omega_\mathrm{s}' - \frac{\Delta\Omega}{2}\right) + 2\gamma_\mathrm{ss} + 2\gamma_\mathrm{ii}}\,, \tag{13}$$

$$\rho_\pm(\omega) = \frac{1}{g}\frac{\chi' A_\mathrm{p0}^2}{\pm 2g + 2\mathrm{i}\left(\omega - \omega_\mathrm{i}' - \frac{\Delta\Omega}{2}\right) + \gamma_\mathrm{ss} + \gamma_\mathrm{ii}}\,. \tag{14}$$

We assume the heat bath to be initially in thermodynamic equilibrium at temperature T, where the occupation number of the k-th mode is governed by Bose-Einstein statistics, i.e.

$$\bar{n}(\omega_k) = \left[\exp\left(\frac{\hbar\omega_k}{k_\mathrm{B}T}\right) - 1\right]^{-1}\,. \tag{15}$$

Furthermore, we assume that photons in different heat bath modes are uncorrelated. The time evolution of the number of photons in the signal mode, which is a measure for the gain of the JTWPA, is represented by the photon number operator $\tilde{\hat{A}}_\mathrm{s}^\dagger\tilde{\hat{A}}_\mathrm{s}$. The gain of the amplifier is hence defined by the ratio of the expected number of photons at the output to the number of photons at the input of the amplifier, i.e.

$$\begin{aligned}
G_\mathrm{signal}^\mathrm{q} = &\left[\left|\left(\cosh(gt) + \frac{-\gamma_\mathrm{ss} + \gamma_\mathrm{ii} + \mathrm{i}\Delta\Omega}{2g}\sinh(gt)\right)\right|^2\right.\\
&\left. + \frac{\left\langle\tilde{\hat{A}}_\mathrm{i0}^\dagger\tilde{\hat{A}}_\mathrm{i0}\right\rangle + 1}{\left\langle\tilde{\hat{A}}_\mathrm{s0}^\dagger\tilde{\hat{A}}_\mathrm{s0}\right\rangle}\left|\frac{\chi' A_\mathrm{p0}^2}{g}\sinh(gt)\right|^2\right]\mathrm{e}^{-(\gamma_\mathrm{ss}+\gamma_\mathrm{ii})t}\\
&+ \frac{1}{\left\langle\tilde{\hat{A}}_\mathrm{s0}^\dagger\tilde{\hat{A}}_\mathrm{s0}\right\rangle}N_\mathrm{noise}(\bar{n}_\mathrm{s}, \bar{n}_\mathrm{i})\,.
\end{aligned} \tag{16}$$

The first line in (16) corresponds to the amplification of the signal photons at the input of the amplifier. The second line describes the converted idler noise, i.e. photons converted from the idler mode to the signal mode. Both contributions experience exponential damping by a factor of $\exp[-(\gamma_\mathrm{ss} + \gamma_\mathrm{ii})t]$. The contributions of the system-bath interaction from (12) are summarized in the function N_noise, which depends on the thermal occupations of the heat bath modes related to the signal and idler frequencies by $\bar{n}_\mathrm{s} = \bar{n}\left(\omega_\mathrm{s}' + \frac{\Delta\Omega}{2}\right)$ and $\bar{n}_\mathrm{i} = \bar{n}\left(\omega_\mathrm{i}' + \frac{\Delta\Omega}{2}\right)$, respectively.

Our expression for the gain (16) extends the theory presented in [18] by considering noise and dissipation. In the limiting case, assuming no coupling between system and heat bath, our expression for the gain is equal to the one obtained in [18].

To highlight the difference between the ideal case, without interaction with the environment, and our model where photons in the signal and idler modes only have a limited lifetime, let us consider the temporal evolution of the signal energy using a JTWPA from literature [18]. The relevant device parameters of the JTWPA are given in the following. The gound capacitance of the $50\,\Omega$ transmission line is $C = 39\,\mathrm{fF}$. The transmission line inductance is assumed to be negligible compared to the Josephson inductance. The capacitance at the junctions is $C_\mathrm{J} = 329\,\mathrm{fF}$. The critical current of the Josephson junctions is given by $I_\mathrm{c} = 3.29\,\mu\mathrm{A}$.

We consider two cases, one where we have interaction with the environment with coupling constants of $\gamma_\mathrm{ss} = \gamma_\mathrm{ii} = 0.5 \times 10^8\,\mathrm{s}^{-1}$ at $4.2\,\mathrm{K}$ liquid Helium temperature and the ideal case without environmental coupling, as given in [18]. Figure 3 shows the temporal evolution of the signal energy for different pump currents, i.e. $I_\mathrm{p} = 0.5 \cdot I_\mathrm{c}$ and $I_\mathrm{p} = 0.7 \cdot I_\mathrm{c}$. The signal frequency is chosen to be $\omega_\mathrm{s} = 2\pi \cdot 5.9\,\mathrm{GHz}$ with a pump frequency of $\omega_\mathrm{p} = 2\pi \cdot 5.97\,\mathrm{GHz}$. The input is assumed to be a single photon. The difference can be explained by the damping in (16).

5 Conclusion

We have introduced noise and dissipation in a circuit quantum electrodynamic model of a Josephson traveling-wave parametric amplifier. The Hamiltonian of the dissipative system was derived from reservoir theory and an approximate analytical solution to the resulting equations of motion

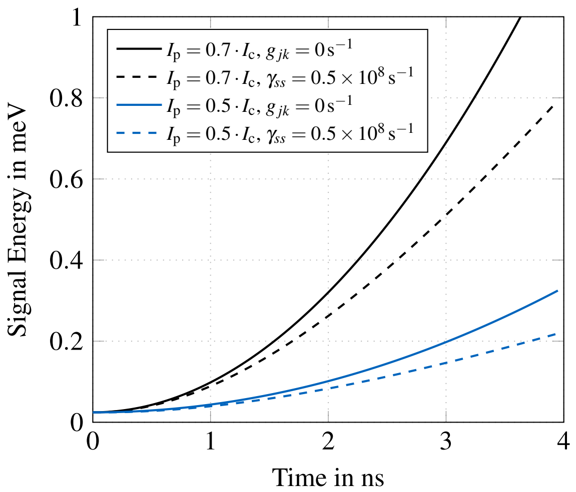

Figure 3. Temporal evolution of the energy in the signal mode for different pump currents I_p, given a single phonton at $t = 0$. Phase matching is not considered here.

was presented. We have discussed the implications of the environmental coupling by using a JTWPA device from the literature. A quantum model for noise and dissipation can enhance the design, as well as the understanding of parametric amplification in the quantum realm.

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