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Nonlinear Coupled Surface TE and Leaky TM Electromagnetic Waves

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Abstract

Nonlinear coupled electromagnetic surface TE and leaky TM wave propagation in a cylindrical nonlinear inhomogeneous metal-dielectric waveguide (Goubau line) with circular cross section is considered. Physical problem is reduced to a nonlinear two-parameter eigenvalue problem for a system of (nonlinear) ordinary differential equations. For the numerical solution, a method based on solving an auxiliary Cauchy problem (the shooting method) is proposed. The coupled TE-TM waves propagating in GL are determined numerically.

1 Introduction

In this paper nonlinear coupled electromagnetic surface TE and leaky TM wave propagation in a perfectly conducting cylinder covered by a concentric dielectric layer, the Goubau line (GL), with Kerr nonlinearity is considered. Here we show (numerically) that in this case new propagation regime exists (to compare see Ref. [1]). In this regime we have sum of two independent waves (surface TE and leaky TM) that create new (nonlinear) polarization (so called coupled TE-TM wave). TE and TM waves that form this nonlinear TE-TM wave we call pseudopolarizations (pseudo-TE or pseudo-TM, respectively). In this case we see that pseudo-TE and pseudo-TM waves can propagate at their own frequencies (ω_E or ω_M respectively) and they have their own propagation constants (γ_E or γ_M for pseudo-TE or pseudo-TM waves, respectively). The physical problem is reduced to a two-parameter eigenvalue problem for Maxwell's equations. The opportunity to consider nonlinear coupled TE-TM wave propagation at two different frequencies was pointed out in Ref. [2].

This problem, as it is known to the authors, has not been considered in literature. This regime is seemed to be interesting for, at least, this model can be applied to study nonlinear interaction of two types of waves (surface TE and leaky TM) at different frequencies. We should note that frequencies ω_E and ω_M do not depend on each other.

Short background of nonlinear guided waves in media with Kerr and Kerr-like nonlinearities is given in Ref.[1]. For purely nonlinear TE and TM waves see Refs. [3]-[8].

2 Statement of the problem

Let us consider three-dimensional space $\mathbb{R}^3$ with a cylindrical coordinate system $O\rho\varphi z$. The space is filled with an isotropic and nonmagnetic medium (it is supposed that everywhere $\mu = \mu_0$, where μ_0 is the permeability of free space) having the constant permittivity $\varepsilon_0 = const$ where ε_0 is the permittivity of free space, without sources. A perfectly conducting cylinder covered by a dielectric layer with a cross-section

$$\Sigma := \{(\rho,\varphi) : r_0 \leqslant \rho \leqslant r, 0 \leqslant \varphi < 2\pi\}$$

and a generating line parallel to the axis Oz is placed in $\mathbb{R}^3$.

The cross section of GL consists of two concentric circles of radii r_0 and r (see Fig. 1): r_0 is the radius of the internal (perfectly conducting) cylinder and $r_0 - r$ is the thickness of the external (dielectric) cylindrical shell.

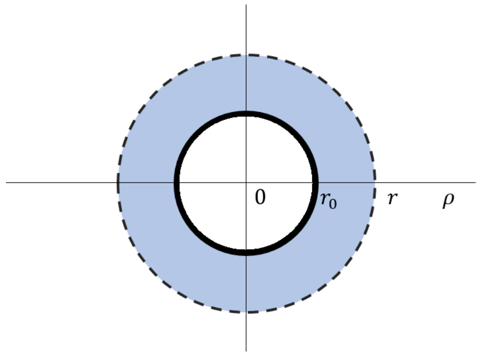

Figure 1. Goubau line; a single wire conductor coated with dielectric material.

In this study, the electromagnetic field is assumed to be a linear combination of monochromatic TE and TM waves (each polarisation propagates at its own frequency and depends harmonically on z with different but coupled propagation constants). In a nonlinear medium such a linear combination is a new type of guided waves, called the coupled TE-TM wave.

Introduce the **E**, **H** in the following way

$$\mathbf{E} = \mathbf{E}_E e^{-i\omega_E t} + \mathbf{E}_M e^{-i\omega_M t},\ \mathbf{H} = \mathbf{H}_E e^{-i\omega_E t} + \mathbf{H}_M e^{-i\omega_M t},$$

where $\mathbf{E}_E$, $\mathbf{E}_M$ and $\mathbf{H}_E$, $\mathbf{H}_M$ are complex amplitudes [9] having the form

$$\begin{aligned} &\mathbf{E}_E = (0, E_\varphi, 0)^\top, \quad \mathbf{E}_M = (E_\rho, 0, E_z)^\top, \\ &\mathbf{H}_E = (H_\rho, 0, H_z)^\top, \quad \mathbf{H}_M = (0, H_\varphi, 0)^\top, \end{aligned} \tag{1}$$

and

$$\begin{aligned} &E_\rho = \mathrm{E}_\rho(\rho) e^{i\gamma_M z},\ E_\varphi = \mathrm{E}_\varphi(\rho) e^{i\gamma_E z},\ E_z = \mathrm{E}_z(\rho) e^{i\gamma_M z}, \\ &H_\rho = \mathrm{H}_\rho(\rho) e^{i\gamma_E z},\ H_\varphi = \mathrm{H}_\varphi(\rho) e^{i\gamma_M z},\ H_z = \mathrm{H}_z(\rho) e^{i\gamma_E z}, \end{aligned}$$

where γ_E, γ_M are unknown positive real (coupled) propagation constants (spectral parameters).

Choosing fields in the form of a linear combination of TE and TM waves (which do not depend on φ coordinate) is caused by the fact that, firstly, for such a dual-frequency field the problem can be solved without additional simplifying assumptions (we mean that all the simplifications are contained in the formula for the permittivity); secondly, monochromatic TE and TM waves in circular cylindrical waveguides are observed experimentally [10, 11] and have important applications.

We assume that permittivity in the entire space has the form $\widetilde{\varepsilon}\varepsilon_0$, where

$$\widetilde{\varepsilon} = \begin{cases} \varepsilon + \alpha|\mathbf{E}|^2, & r_0 \leqslant \rho \leqslant r, \\ 1, & \rho > r, \end{cases} \tag{2}$$

α is a real positive constant and $\varepsilon(\rho)$ is a continuous function for $\rho \in [r_0, r]$.

The problem of coupled TE-TM wave propagation along a GL, called problem P, is reduced to determine discrete pairs $(\gamma_E, \gamma_M) = (\widehat{\gamma}_E, \widehat{\gamma}_M)$ for which there are vectors (1) satisfying Maxwell's equations, the continuity condition for the tangential components of the field at the interface $\rho = r$ and the radiation conditions at infinity, which will be formulated and discussed below. The tangential components of electric field $\mathbf{E}$ vanishes on a perfectly conducting boundary $\rho = r_0$.

Definition 1. A pair of numbers $(\widehat{\gamma}_E, \widehat{\gamma}_M)$ is called coupled propagation constants, if $(\gamma_E, \gamma_M) = (\widehat{\gamma}_E, \widehat{\gamma}_M)$ is a solution to problem P.

Since we look for classical solutions, it is assumed that the field components are continuous and have all required derivatives for $\rho \in [r_0, r]$.

Substituting (1) in Maxwell's equations, after simplifying, we obtain

$$\begin{cases} \gamma_M^2 \mathrm{E}_\rho + \gamma_M (i\mathrm{E}_z)' = \omega_M^2 \mu_0 \varepsilon_0 \widetilde{\varepsilon} \mathrm{E}_\rho, \\ \gamma_E^2 \mathrm{E}_\varphi - \left(\rho^{-1}(\rho \mathrm{E}_\varphi)'\right)' = \omega_E^2 \mu_0 \varepsilon_0 \widetilde{\varepsilon} \mathrm{E}_\varphi, \\ -\gamma_M \rho^{-1}(\rho \mathrm{E}_\rho)' - \rho^{-1}(\rho (i\mathrm{E}_z)')' = \omega_M^2 \mu_0 \varepsilon_0 \widetilde{\varepsilon}(i\mathrm{E}_z), \end{cases} \tag{3}$$

where $\mathrm{H}_\rho = -\gamma_E(\omega_E \mu_0)^{-1} \mathrm{E}_\varphi$, $\mathrm{H}_z = (i\omega_E \mu_0)^{-1} \rho^{-1} (\rho \mathrm{E}_\varphi)'$ and$\mathrm{H}_\varphi = (i\omega_M \mu_0)^{-1}(i\gamma_M \mathrm{E}_\rho - \mathrm{E}_z')$.

Introducing the notation

$$u_1(\rho) := \mathrm{E}_\rho(\rho),\ u_2(\rho) := \mathrm{E}_\varphi(\rho),\ u_3(\rho) := i\mathrm{E}_z(\rho),$$

where u_i depends on both ρ and γ_E, γ_M (further we omit the arguments), we rewrite (3) in the form

$$\begin{cases} \gamma_M^2 u_1 + \gamma_M u_3' = \nu_M^2 \widetilde{\varepsilon} u_1, \\ \left(\rho^{-1}(\rho u_2)'\right)' - \gamma_E^2 u_2 = -\nu_E^2 \widetilde{\varepsilon} u_2, \\ \gamma_M \rho^{-1}(\rho u_1)' + \rho^{-1}(\rho u_3')' = -\nu_M^2 \widetilde{\varepsilon} u_3, \end{cases} \tag{4}$$

where $\nu_E^2 := \omega_E^2 \mu_0 \varepsilon_0$ and $\nu_M^2 := \omega_M^2 \mu_0 \varepsilon_0$.

For $\rho > r$, we have $\widetilde{\varepsilon} = 1$. System (4) gives

$$\begin{cases} \gamma_M^2 u_1 + \gamma_M u_3' = \nu_M^2 u_1, \\ \left(\rho^{-1}(\rho u_2)'\right)' - \gamma_E^2 u_2 = -\nu_E^2 u_2, \\ \gamma_M \rho^{-1}(\rho u_1)' + \rho^{-1}(\rho u_3')' = -\nu_M^2 u_3. \end{cases}$$

The solution of this system has the form

$$\begin{cases} u_1(\rho) = -\gamma_M \kappa_M^{-1} \left(\widetilde{C}_M K_0'(\kappa_M \rho) + C_M I_0'(\kappa_M \rho) \right), \\ u_2(\rho) = \widetilde{C}_E I_1(\kappa_E \rho) + C_E K_1(\kappa_E \rho), \\ u_3(\rho) = \widetilde{C}_M K_0(\kappa_M \rho) + C_M I_0(\kappa_M \rho), \end{cases} \tag{5}$$

where $\kappa_M^2 := \gamma_M^2 - \nu_M^2$, $\kappa_E^2 := \gamma_E^2 - \nu_E^2$; I and K are the modified Bessel function (Infield and Macdonald function) and $\widetilde{C}_E$, $\widetilde{C}_M$, C_M and C_E are constants.

Remark 1. Classification of waves as surface or leaky depends on the behaviour in ρ-direction (see, for example [10, 11]). The *surface* wave is such that $u(\rho) \to 0$, $\rho \to \infty$. The *leaky* wave is such that $u(\rho) \to \infty$, $\rho \to \infty$.

We will consider coupled surface TE and leaky TM waves. The solution $C_E K_1(\kappa_E \rho)$ determines *surface TE-polarized waves* decreasing at infinity because $K_1(\kappa_E \rho) \to 0$ as $\rho \to \infty$ (see [12]). The solution $C_M I_0(\kappa_M \rho)$ determines *leaky TM-polarized waves* increasing at infinity because $I_0(\kappa_M \rho) \to \infty$ as $\rho \to \infty$ (see [12]).

In view of the conditions at infinity, solutions (5) takes the form

$$\begin{cases} u_1(\rho) = -\gamma_M \kappa_M^{-1} C_M I_1(\kappa_M \rho), \\ u_2(\rho) = C_E K_1(\kappa_E \rho), \\ u_3(\rho) = C_M I_0(\kappa_M \rho). \end{cases} \tag{6}$$

For $r_0 \leqslant \rho \leqslant r$, we have $\widetilde{\varepsilon} = \varepsilon + \alpha|\mathbf{u}|^2$, where $|\mathbf{u}|^2 = u_1^2 + u_2^2 + u_3^2$ and $u = (u_1, u_2, u_3)^\top$. System(4) gives the system of nonlinear equations

$$\begin{cases} \gamma_M^2 u_1 + \gamma_M u_3' = \nu_M^2 (\varepsilon + \alpha|\mathbf{u}|^2) u_1, \\ \left(\rho^{-1}(\rho u_2)'\right)' - \gamma_E^2 u_2 = -\nu_E^2 (\varepsilon + \alpha|\mathbf{u}|^2) u_2, \\ \gamma_M \rho^{-1}(\rho u_1)' + \rho^{-1}(\rho u_3')' = -\nu_M^2 (\varepsilon + \alpha|\mathbf{u}|^2) u_3. \end{cases} \tag{7}$$

Tangential components of electromagnetic field are known to be continuous at the interfaces. In this case the tangential components are E_φ, E_z, H_φ, and H_z. Since the component E_ρ is a normal component and therefore it has a finite jump on the interface $\rho = r$, then the quantity $\widetilde{\varepsilon}E_\rho$ is continuous on the interface $\rho = r$. The tangential components of electric field $\mathbf{E}$ are vanishes on a perfectly conducting boundary $\rho = r_0$. This results in the following boundary and transmission conditions

$$u_2|_{\rho=r_0} = 0,\ u_3|_{\rho=r_0} = 0, \tag{8}$$

and

$$[\widetilde{\varepsilon}u_1]|_r = 0,\ [u_2]|_r = 0,\ [u_2']\big|_r = 0,\ [u_3]|_r = 0, \tag{9}$$

where $[f]|_r = \lim\limits_{\rho\to r-0} f(\rho) - \lim\limits_{\rho\to r+0} f(\rho)$.

Thus problem P is reduced to finding pairs $(\gamma_E, \gamma_M) = (\widehat{\gamma}_E, \widehat{\gamma}_M)$ such that for fixed and non-zero values of C_E, C_M, where without loss of generality both constants are positive, there exists a vector $\mathbf{u} = (u_1, u_2, u_3)^\top$ having components satisfying (7); these components must satisfy boundary conditions (8), transmission conditions (9) involving boundary values of solutions (6).

From the mathematical standpoint, the problem P is a nonlinear two-parameter eigenvalue problem with boundary conditions defined by (8), (9), and (6); pairs $(\widehat{\gamma}_E, \widehat{\gamma}_M)$ are coupled eigenvalues.

3 Numerical method

To determine the approximate the pair of propagation constants of problem P, we will use the shooting method.

Consider the Cauchy problem for the system of equations

$$\begin{cases} u_1' = \dfrac{\varepsilon' + 2\alpha\left(u_2u_4 - \rho^{-1}u_2^2 - \gamma_M u_1 u_3\right)}{\varepsilon + \alpha\left(2u_1^2 + |\mathbf{u}|^2\right)} - \\ \quad - \dfrac{\rho^{-1}u_1 + \gamma_M u_3 + 2\alpha k_M^2 \gamma_M^{-1} u_1^2 u_3}{\varepsilon + \alpha\left(2u_1^2 + |\mathbf{u}|^2\right)}(\varepsilon + \alpha|\mathbf{u}|^2), \\ u_2' = u_4 - \rho^{-1}u_2, \\ u_3' = -\gamma_M u_1 + \gamma_M^{-1} k_M^2 (\varepsilon + \alpha|\mathbf{u}|^2) u_1, \\ u_4' = \gamma_E^2 u_2 - k_E^2(\varepsilon + \alpha|\mathbf{u}|^2) u_2 \end{cases} \tag{10}$$

with the following initial conditions

$$\begin{cases} u_1(r) = x, \\ u_2(r) = C_E K_1(\kappa_E r), \\ u_3(r) = C_M I_0(\kappa_M r), \\ u_4(r) = -C_E K_0(\kappa_E r), \end{cases} \tag{11}$$

where value x is determined as solution of following cubic equation

$$Ax^3 + Bx + C = 0,$$

where $A = \alpha$, $B = \varepsilon(r) + \alpha\left(C_E^2 K_1^2(\kappa_E r) + C_M^2 I_0^2(\kappa_M r)\right)$, and $C = -\gamma_M \kappa_M^{-1} C_M I_1(\kappa_M r)$.

Note that, unlike problems for linear media, in the case of nonlinear media, the propagation constants (eigenvalues) depend on the field amplitude.

To justify the solution technique, we use classical results of the theory of ordinary differential equations concerning the existence and uniqueness of the solution to the Cauchy problem and continuous dependence of the solution on parameters.

Using the transmission condition on the boundary r we obtain the following system of dispersion equations

$$\begin{cases} u_2(r_0,\ \gamma_E, \gamma_M) = 0, \\ u_3(r_0,\ \gamma_E, \gamma_M) = 0, \end{cases} \tag{12}$$

where quantities $u_2(r_0)$ and $u_3(r_0)$ are obtained from the solution to the Cauchy problem.

To determine the coupled propagation constants we solve numerically equations (12). Their solutions are curves on the $O\gamma_E\gamma_M$ plane shown in Figs. 2 and 3 in red and blue. The points of intersection of these curves are coupled propagation constants. Grey lines corresponds to "purely" linear surface TE (vertical lines) and leaky TM (horizontal lines) waves. The parameter values used in calculations are in the figure captions. A detailed description of the algorithm can be found in [13].

For a sufficiently small nonlinearity coefficient α, the coupled eigenvalues are found near the points of intersection of the vertical and horizontal segments ("purely" linear TE and TM waves). If the nonlinearity coefficient is less than some critical value, there are at least 4 pairs (Fig. 2).

From Fig. 3 we can see that for large value of nonlinearity coefficient α exists more then 6 pairs of coupled eigenvalues; 4 of the pairs are very close to the solutions of the linear problems (compare with Fig. 2); others pairs are purely nonlinear coupled eigenvalues, they have no connections with the solution of the linear problems.

4 Conclusion

Using the specially developed numerical method we calculated propagation constants of the coupled surface TE and leaky TM wave. For the specific values of parameters the coupled eigenvalues are determined which are close to the solutions of the corresponding linear problems, as well as the new solutions. Whether these mathematically predicted propagation regime really exist is a hypothesis that can be proved or disproved in an experiment.

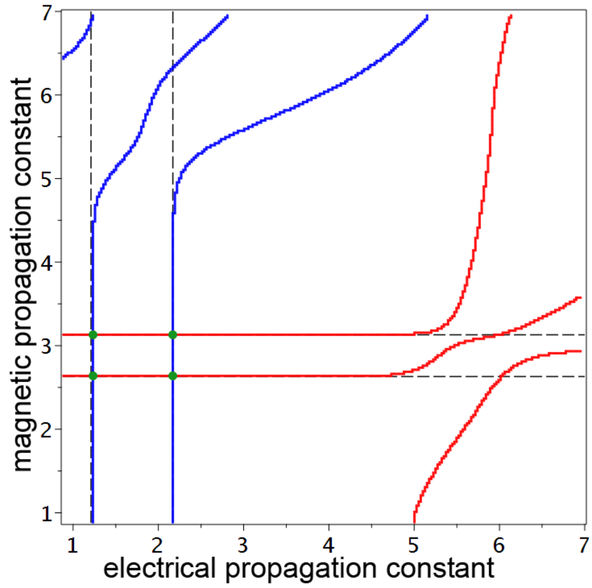

Figure 2. $\varepsilon = \varepsilon_c + \rho$; $\varepsilon_c = 11.7$ (Si); $r_0 = 2$ mm and $r = 4.2$ mm, $C_E = C_M = 1$; $\omega_E = 30$ GHz, $\omega_M = 40$ GHz; $\alpha = 10^{-6}$ mmV^{-1}.

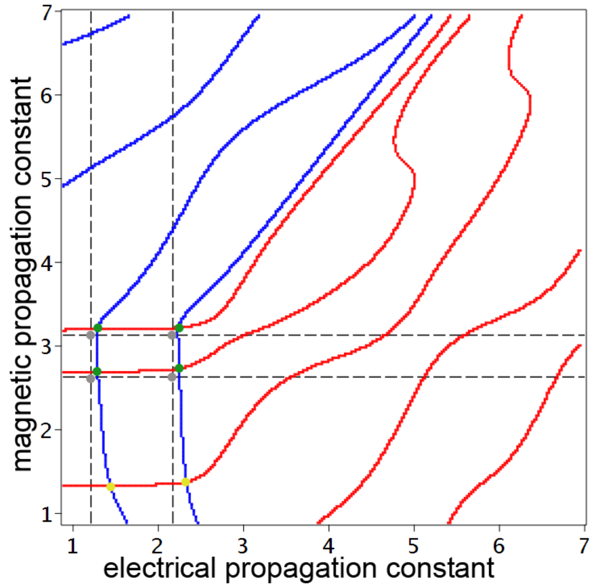

Figure 3. $\varepsilon = \varepsilon_c + \rho$; $\varepsilon_c = 11.7$ (Si); $r_0 = 2$ mm and $r = 4.2$ mm, $C_E = C_M = 1$; $\omega_E = 30$ GHz, $\omega_M = 40$ GHz; $\alpha = 1$ mmV^{-1}.

5 Acknowledgements

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