Electrical Response of Radially Anisotropic Spheroidal Scatterers

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Abstract

This contribution describes a spheroidal inclusion which consists of anisotropic material. The electrical response is solved for a static and uniform excitation field. The anisotropy of the inclusion is assumed to be spheroidally radial, which means that a radially oriented electric field sees a different material parameter as compared to a tangentially oriented one but the field makes no difference between the separate tangential orientations. The goal of this contribution is to outline the analytical procedure that gives the electrical response of the inclusion.

1 Introduction

Spheroidal structures occupy an important position in the electromagnetic field theory. More general a shape than a simple sphere, spheroids closely resemble many structures found in nature while still retaining enough mathematical simplicity to be thoroughly analyzed. Nature provides us with plentiful examples of spheroidal structures from tiny falling water droplets to vast planets and moons on their orbits. Given the spheroid's wide range of applications, it is fortunate that from the mathematical perspective this shape falls within the small category of canonical shapes. This means that we can device coordinate systems that introduce the spheroidal shape by their coordinate isosurfaces and also allow us to separate the Laplace equation. There are two of these coordinate systems, one that suits prolate spheroidal inclusions and another that suits oblate spheroidal inclusions. These are the two spheroidal coordinate systems among the number of eleven orthogonal second degree coordinate systems described in Ref. [1]. The spheroidal coordinate system allows us to solve a wide range of problems in the electromagnetic field theory.

If we confine our attention to slowly varying electric fields and assume that the wavelength is several orders of magnitude greater than the dimension of the spheroidal inclusion, we find that the properties of a homogeneous spheroidal inclusion are well known [2]. Given homogeneity, the existing body of scientific knowledge covers even the broader class of ellipsoidal inclusions. However, an important category of inhomogeneous spheroidal inclusions has remained open to analysis. These scatterers – the topic of the present research – consist of tightly stratified spheroidal layers combined into an onion-like structure. We refer to these scatterers as *radially anisotropic* (RA) spheroids in accordance with established terminology for similar spherical structures [3]. The RA spheroid can be thought of as an onion that has been perturbed from its spherical symmetry either by elongating it into a prolate spheroid or by compressing it into an oblate spheroid.

The qualitative description of the inclusion provides an intuition of the scattering problem. However, to solve the scattering problem, one needs to look at the inclusion from a mathematical point of view. It is, therefore, necessary to describe the inclusion with material parameters, which by their very nature entail an averaging process over the material. In the present case, averaging requires us to forget the individual onion layers and to consider instead the material as locally homogeneous but anisotropic. The only pertinent effect that the onion layers have to the present study is that the material parameter seen by the electric field depends on the field orientation with respect to the layers. It consequently becomes necessary to replace the scalar material parameter ϵ of an isotropic inclusion with a dyadic parameter better suited for an anisotropic material. If $\mathbf{u}_\eta$ is the unit vector that points to the spheroidally radial direction, the permittivity dyadic reads

$$\bar{\bar{\epsilon}} = \epsilon_r \mathbf{u}_\eta \mathbf{u}_\eta + \epsilon_t (\bar{\bar{I}} - \mathbf{u}_\eta \mathbf{u}_\eta) \tag{1}$$

It is assumed that the inclusion lies in a homogeneous background, permeated by a uniform and static electric excitation field $\mathbf{E}^{\mathrm{p}}$. In that situation, we seek to concisely describe the electric response of the inclusion. As the excitation field affects the inclusion, the material inside the inclusion becomes polarized giving

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rise to a perturbation electric field $\mathbf{E}^{\mathrm{s}}$. Far away from the inclusion, this perturbation field approximately takes the form of a dipole field $\mathbf{E}^{\mathrm{s}} \approx \mathbf{E}_{\mathrm{d}}$. Because the far field resembles a dipole field, an equivalent field could be produced by replacing the original inclusion by an ideal dipole. The magnitude of this replacement dipole in proportion to the magnitude of the excitation field characterizes the scattering properties of the inclusion in the far region. In particular, we use the polarizability dyadic $\bar{\bar{\alpha}}$ defined by

$$\mathbf{p} = \bar{\bar{\alpha}} \cdot \mathbf{E}^{\mathrm{p}} \tag{2}$$

to describe the linear relation between the excitation field and the dipole moment. The polarizability $\bar{\bar{\alpha}}$ can be further reduced into the normalized polarizability

$$\bar{\bar{\alpha}}_{\mathrm{n}} = \frac{\bar{\bar{\alpha}}}{\epsilon_0 V} \tag{3}$$

to obtain a dimensionless quantity.

Although there are three dimensions of freedom in the excitation field $\mathbf{E}^{\mathrm{p}}$ the polarizability dyadic has only two. The missing dimension of freedom results from the symmetry of the inclusion. When a sphere is transmutated into a spheroid the rotational symmetry with respect to the origin is compromized, but there remains a rotational symmetry with respect to an axis, which we take to be the z-axis of the Cartesian coordinate system. The rotational symmetry implies that it suffices to consider two polarizations of the excitation field, one parallel to the axis of symmetry $\mathbf{E}^{\mathrm{p}}_{\parallel}$ and another one that is perpendicular $\mathbf{E}^{\mathrm{p}}_{\perp}$ and can be selected arbitrarily from the xy-plane. With these two, an arbitrary polarization can be expressed as a linear combination. If we denote the inclusion's polarizability in a parallel field by $\alpha_{\parallel}$ and that in a perpendicular field by $\alpha_{\perp}$, we can combine the two to give the polarizability dyadic

$$\bar{\bar{\alpha}} = \alpha_{\parallel} \mathbf{u}_z \mathbf{u}_z + \alpha_{\perp} (\bar{\bar{I}} - \mathbf{u}_z \mathbf{u}_z) \tag{4}$$

2 Spheroidal coordinates

The coordinate system that proves most useful for the study of spheroidal inclusions is the spheroidal coordinate system (η, θ, ψ), described in Ref. [1]. The spheroidal coordinate system comes in two formats, one for prolate spheroids and another for oblate spheroids. Here, we confine our attention to prolate spheroids because the two cases are essentially similar. The prolate spheroidal coordinates (η, θ, ψ) can be defined by their relation to the Cartesian coordinates

$$\begin{cases} x = a \sinh \eta \sin \theta \cos \psi \\ y = a \sinh \eta \sin \theta \sin \psi \\ z = a \cosh \eta \cos \theta \end{cases} \tag{5}$$

where a is the distance of a focal point from the center. If the polar radius of the inclusion is b and the equatorial radius is c, the relation between the radii and the focal distance is $a = (b^2 - c^2)^{1/2}$. The virtue of the prolate spheroidal coordinate system is that each of the isocoordinate surfaces $\eta = \mathrm{const}$ takes the shape of a prolate spheroid. The coordinate system allows us to separate the Laplace equation. The Laplace equation, however, is only applicable in an isotropic and homogeneous medium. When we consider an anisotropic and inhomogeneous medium, the equation becomes

$$\nabla \cdot \bar{\bar{\epsilon}} \cdot \nabla \phi = 0 \tag{6}$$

Fortunately this equation also is separable in the spheroidal coordinate systems.

In prolate spheroidal coordinates the generalized Laplace equation (6) reads

$$\frac{1}{a^2\left(\sinh^2 \eta + \sin^2 \theta\right)} \left\{ \epsilon_\eta \frac{\partial^2 \phi}{\partial \eta^2} + \epsilon_\eta \coth \eta \frac{\partial \phi}{\partial \eta} + \epsilon_\theta \frac{\partial^2 \phi}{\partial \theta^2} + \epsilon_\theta \cot \theta \frac{\partial \phi}{\partial \theta} \right\} + \epsilon_\psi \frac{1}{a^2 \sinh \eta \sin \theta} \frac{\partial^2 \phi}{\partial \psi^2} = 0 \tag{7}$$

When we substitute $\phi(\mathbf{r}) = H(\eta)\Theta(\theta)\Psi(\psi)$ the equation separates into three parts. Then the complete set of solutions presents itself. It is, however, safe to disregard most of the solutions. The only solutions of (7) pertinent to the present scattering problem are the ones whose angular part $\Theta(\theta)\Psi(\psi)$ agrees with that of the excitation field. That consideration leaves us with two solutions for the potential inside the inclusion ϕ_{in} and the perturbation potential ϕ^{s}. When the excitation field is parallel, the solution is

$$\begin{cases} \phi^{\text{s}}(\mathbf{r}) = AQ_1(\cosh\eta)\cos\theta \\ \phi_{\text{in}}(\mathbf{r}) = BP_\lambda(\cosh\eta)\cos\theta \end{cases} \tag{8}$$

and when the excitation field is perpendicular, the solution is

$$\begin{cases} \phi^{\text{s}}(\mathbf{r}) = AQ_1^1(\cosh\eta)\sin\theta\cos\psi \\ \phi_{\text{in}}(\mathbf{r}) = BP_\lambda^\mu(\cosh\eta)\sin\theta\cos\psi \end{cases} \tag{9}$$

where we denote the two Legendre functions by P and Q. Similarly, in the oblate spheroidal coordinate system we differentiate between the parallel case and the perpendicular one.

To find the perturbation potential ϕ^{s} and thereby the normalized polarizability $\bar{\bar{\alpha}}_{\text{n}}$, it suffices to find the constant A by applying the boundary conditions $\phi_{\text{in}} = \phi_{\text{out}}$ and $\epsilon_r\partial_\eta\phi_{\text{in}} = \partial_\eta\phi_{\text{out}}$. In the case of parallel polarization, the solution for the normalized polarizability $\alpha_{\text{n},\parallel}$ reads

$$\alpha_{\text{n},\parallel} = \frac{a^3}{c^2b}\frac{\left(1-x_0^2\right)P_\lambda(x_0) - (\lambda+1)\epsilon_\eta x_0\left[x_0P_\lambda(x_0) - P_{\lambda+1}(x_0)\right]}{2P_\lambda(x_0)\left(x_0Q_1(x_0) - Q_2(x_0)\right) - (\lambda+1)\epsilon_\eta Q_1(x_0)\left[x_0P_\lambda(x_0) - P_{\lambda+1}(x_0)\right]} \tag{10}$$

where the parameters λ and x_0 are

$$\lambda = -\frac{1}{2} + \sqrt{2\frac{\epsilon_t}{\epsilon_r} + \frac{1}{4}} \tag{11}$$

and

$$x_0 = \frac{b}{a} = \frac{b}{\sqrt{b^2 - c^2}} \tag{12}$$

3 Results

The analytical formula (10) for the RA spheroid covers a wide range of geometrical shapes. We do, however, adopt the prolate spheroidal coordinates to derive the analytical formula. This coordinate system is applicable only when the inclusion's polar radius b is strictly greater than the equatorial radius c. Similarly, for the oblate case we need to assume $b < c$ when we derive the polarizability.

To validate our equation against established results, we need to overcome this apparent restriction. The restriction recedes, when we note that the equation (10) makes no explicit reference to coordinates. The equation remains valid even when the coordinates become obsolete. In particular, the results can be obtained for a vanishingly thin stick $b \gg c$, a vanishingly thin disk $c \gg b$, and a sphere $b = c$. Because the polarizability for an RA sphere is known [3,4], and the polarizabilities for RA stick and RA disk can be easily divined, the polarizability equation can be validated in three special cases.

To discover the special cases, we employ a limiting procedure on Eq. (10). In this procedure, the parameter x_0 obtains values close to positive infinity $x_0 \to \infty$ or $x_0 \to 1+$. When x_0 approaches positive infinity, the inclusion resembles an RA sphere. If, in contrast, the parameter x_0 acquires values $x_0 \approx 1$ the inclusion resembles a stick. The third special case, the disk, emerges as a limiting case of an oblate spheroid. It requires a different formula but can be analyzed similarly.

The three special cases are suitable for validation because a comparison can be found in the existing body of scientific knowledge. However, the analytical method of present study allows us explore a wider range of inclusions. The results, seen in Fig. 1, show that in case of prolate spheroid the two permittivity components contribute on unequal basis. As one would expect, the tangential permittivity component, which more readily meets the excitation electric field, affects the polarizability more forcefully than the less pertinent radial component. The disparity between the two components is less striking in the case of a sphere because the more symmetric shape brings the radial component to bear on the excitation field.

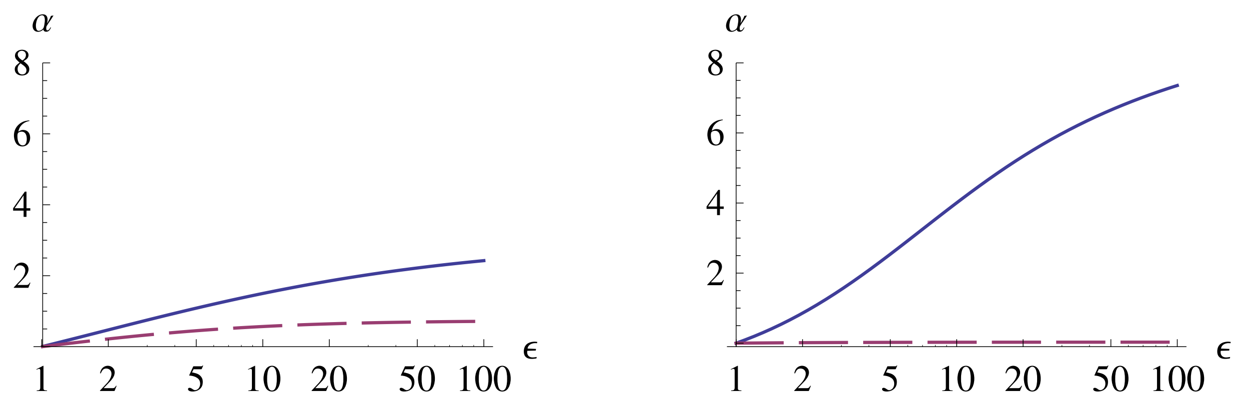

Figure 1: Axial polarizabilies of an anisotropic sphere (left) and an anisotropic prolate spheroid with axis ratio 3 : 1 (right). The solid line represents varying ϵ_t with $\epsilon_r = 1$. The dashed line represents varying ϵ_r with $\epsilon_t = 1$.

4 Conclusion

This contribution introduced an analytical method to analyze the electrostatic scattering problem involving an RA spheroid as an inclusion. A procedure was described that first solves the Laplace equation in the suitable coordinate system and then combines the pertinent solutions of two different regions to produce a physically viable electric potential. Particular attention was given to the perturbation potential ϕ^{s} and normalized polarizability $\bar{\bar{\alpha}}$. The polarizability dyadic was decomposed into two components $\alpha_{\|}$ and $\alpha_{\perp}$ according to the rotational symmetry of the scattering problem. For the interest of brevity, an explicit solution was presented only in the case of parallel polarization and prolate spheroid. The same method, however, applies for both polarizations and both types of spheroids.

The contribution also sketched a method for validating the results. It outlined a limiting procedure that reduces the general formula into well-known special cases. The three cases were stick, disk, and sphere. A good agreement was found in all three test cases.

Even if geometrically simple, the spheroidal shape can be found in nature where it abounds in its perennial existence. The study of spheroidal scatterers offers insights into the workings of nature. By its tractable mathematical character, the spheroids help us to push the frontiers of knowledge towards an increasingly diverse set of discoveries.

5 References

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