# Asymptotic expansion of problem solution of spherical wave diffraction on wedge 

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#### Abstract

This work is dedicated to the study the of an electromagnetic field behavior in a wedge-shaped region with wedge opening angle $2 \Phi$ close to the straight angle, $2 \Phi \rightarrow \pi$. This geometry is a useful model for applications in radionavigation aids, in cellar communication. In this work we present the asymptotic expansions of the Helmholtz equation solution with the right part corresponding to the electric dipole. Dissipation field of spherical wave at a wedge with perfectly conducting face, with the wedge opening angle close to the strait angle is described by the sum of the fields of four half-shadow waves and an edge wave field. Half-shadow waves are calculated by Macdonalds integrals. Computations are confirmed by measurements on the wedge model.


## 1. Introduction

The investigation of electromagnetic wave diffraction on wedge-shaped structures has very important applied meaning for electromagnetic field calculation, for example, in the radio-navigation aids [1], in cellular communications and in other cases [2]. Originally the rigorous solution of the diffraction problem for an infinitely thin perfectly conducting half-plane was found by Sommerfeld [3]. At present number of rigorous results in diffraction on thin screens is large. The basis for getting the majority of them presents the defined diffraction field in the form of Sommerfeld contour integral. Numerous attempts to receive short-wave asymptotic expansions of the considered problem solution were made by different authors. In this work the case is considered an important for practical applications when the wedge opening angle value is close to the straight angle, $2 \Phi \rightarrow \pi$. This case has the following peculiarity: two poles turn out to be at a short distance in the vicinity of the saddle point on the standard integrating contour.

## 2. Problem statement

In the present work the derivation of asymptotic expansions is given with $k \rightarrow \infty$ of Helmholtz equation solution

$$
\begin{equation*}
\left(\Delta+\kappa^{2}\right) \vec{\Pi}(\vec{r})=-4 \pi \vec{P} \delta\left(\vec{r}-\vec{r}_{0}\right) \tag{1}
\end{equation*}
$$

$0 \leq \arg k \leq \pi$ in the wedge-shaped region $\mathrm{D}(r=(\rho, \varphi, z), 0<\rho<\infty,|\varphi|<\Phi,|z|<\infty)$ of three-dimensional space, $(\rho, \varphi, z)$ - cylindrical coordinates, where $-\pi<\varphi \leq \pi$.
$\vec{\Pi}(\rho, \varphi, z)$ - Green vector function (Hertz vector), where $\vec{\Pi}(\rho, \varphi, z)$ satisfies edge conditions with $\varphi= \pm \Phi$ :

$$
\begin{equation*}
\left[\vec{n}_{ \pm \Phi}, \vec{\Pi}(\rho, \pm \Phi, z)\right]=0,\left(\partial / \partial n_{ \pm \Phi}\right)\left[\vec{n}_{ \pm \Phi}, \vec{\Pi}(\rho, \pm \Phi, z)\right]=0 \tag{2}
\end{equation*}
$$

and some additional conditions on the wedge edge.
During construction of short-wave asymptotic expansions of the considered problem we proceed from the wellknown integral representations of these solutions [4]. Namely, if to designate $G(\alpha)=\exp [i k R(\alpha)] / R(\alpha)$, where $R(\alpha)=\sqrt{\rho^{2}+\rho_{0}^{2}+\left(z-z_{0}\right)^{2}-2 \rho \rho_{0} \cos \alpha}$, then the solution can be represented in the form:

$$
\begin{equation*}
\vec{\Pi}(\vec{r})=\frac{1}{2 \pi i} \int_{\gamma} G(\alpha) \vec{s}(\alpha+\varphi) d \alpha \tag{3}
\end{equation*}
$$

where $\gamma$-Sommerfeld contour of integration, is shown in Fig. 1 and consists of two loops $\gamma_{1}$ and $\gamma_{2}$ on the complex plane $\alpha$; (the contour $\gamma$ is symmetrical relative to the coordinate origin). In Fig. 1 the kernel $G(\alpha)$ branching points $\alpha=2 \pi n \pm i c$ are shown as well with vertical cuts approaching infinity.


Fig. 1. Sommerfeld contour


Fig. 2. Measurements scheme

$$
\begin{align*}
& c=\operatorname{arc}\left(R_{1} / \sqrt{2 \rho \rho_{0}}\right), \quad R_{1}=R(\pi / 2)=\sqrt{\rho^{2}+\rho_{0}^{2}+\left(z-z_{0}\right)^{2}}: \\
& \vec{s}(\alpha)=\frac{\pi P}{4 \Phi}\left[\vec{e}_{\alpha-\varphi_{0}} \operatorname{ctg} \frac{\pi}{4 \Phi}\left(\alpha-\varphi_{0}\right)-\vec{e}_{\alpha+\varphi_{0}-2 \Phi} \operatorname{ctg} \frac{\pi}{4 \Phi}\left(\alpha+\varphi_{0}-2 \Phi\right)\right] \tag{4}
\end{align*}
$$

Here $P=|\vec{P}|$, and unit orthogonal orts $\vec{e}_{1}, \vec{e}_{2}, \vec{e}_{3}$ such that ort $\vec{e}_{1}$ is directed along the axis $\mathrm{z},\left(\vec{e}_{1}=\vec{k}\right), \vec{e}_{2}$ is defined from the terms $\vec{P}=P\left(\cos \theta_{0} \vec{e}_{1}+\sin \theta_{0} \vec{e}_{2}\right),\left(\theta_{0}-\right.$ vector $\vec{P}$ tilt angle to the axis $\mathrm{z}-$ the wedge edge $)$, where $\vec{e}_{2}=\cos \psi \vec{i}+\sin \psi \vec{j} \quad\left(\vec{i}, \vec{j}, \vec{\kappa} \quad-\quad\right.$ unit vectors of the axes $\mathrm{x}, \quad \mathrm{y}, \mathrm{z}$ and $\vec{e}_{3}=\left[\vec{e}_{1}, \vec{e}_{2}\right]$, $\vec{e}_{\alpha}=\cos \theta_{0} \vec{e}_{1}+\sin \theta_{0}\left(\cos \alpha \vec{e}_{2}+\sin \alpha \vec{e}_{3}\right)$.

## 3. Problem solution

Numerous attempts to receive asymptotic expansions of the considered problem solution were numerously performed by different authors. These attempts were based on saddle points method. To make things clear suppose that $\pi / 3<\Phi<\pi$. This case covers an important for practical applications situation when the wedge opening angle $2 \Phi$ is close to $\pi$ in its value. For simplicity, we assume that the electric dipole is parallel to the wedge edge. Then in the plane perpendicular to the wedge edge Hertz vector has the only component $\Pi_{k}(\rho, \varphi)$, parallel to the wedge edge. Now we complement the contour $\gamma$ up to the closed one by the contours $c_{1}$ and $c_{2}$ (Fig. 1). The integral around the closed contour is estimated by the residue theorem and gives the field geometric optics part. The integrals of $c_{1}$ and $c_{2}$ contours give the field diffraction part. The field diffraction part is presented in the form of the asymptotic sequence expansion $\left(f_{m}\right)_{m \in N}$ when $\left(\kappa \rho \rho_{0} / R_{0}\right) \rightarrow \infty$

$$
\begin{gather*}
f_{m}=\frac{(-1)^{m} i \Gamma(m+1 / 2)}{2} \sqrt{\frac{\pi}{\rho \rho_{0}}}\left(\frac{R_{0}}{\kappa \rho \rho_{0}}\right)^{m} H_{m}^{(1)}\left(\kappa R_{0}\right),  \tag{5}\\
R_{0}=R(\pi)=\rho+\rho_{0} . \tag{6}
\end{gather*}
$$

Summing up the geometric optics fields and the field diffraction part we get:

$$
\begin{equation*}
\Pi_{\kappa}\left(\rho, \rho_{0}, \varphi, \varphi_{o}\right) \sim P\left\{-M\left[-\left(\varphi-\varphi_{1}\right)\right]+M\left[-\left(\varphi-\varphi_{2}\right)\right]+M\left(\varphi-\varphi_{3}\right)-M\left(\varphi-\varphi_{4}\right)+C\left(\rho, \rho_{0}, \varphi, \varphi_{o}\right)\right\} \tag{7}
\end{equation*}
$$

where

$$
\begin{gather*}
M(\beta)=\frac{i k}{2} M_{1}\left[\frac{2 \sqrt{\rho_{0} \rho}}{R(\pi-\beta)} \sin \frac{\beta}{2}, k R(\pi-\beta)\right]  \tag{8}\\
R(\alpha)=\sqrt{\rho_{0}^{2}+\rho^{2}-2 \rho_{0} \rho \cos \alpha} \tag{9}
\end{gather*}
$$

$M_{m}(x, y)=\int_{-\infty}^{\operatorname{Arsh} x} H_{m}^{(1)}(y \cdot \operatorname{ch} \xi) \frac{d \xi}{(\operatorname{ch} \xi)^{m-1}}-$ Macdonald integral of m order.
Here we name the fields described by $M(\beta)$ functions as half-shadow fields. $C\left(\rho, \rho_{0}, \varphi, \varphi_{0}\right)$ - the item describing some component of the edge wave. It is not presented here due to the bulky expression.

## 4. Experimental results

The wedge model was made with the help of flat aluminium alloy sheets. The edge $\varphi=+\Phi$ dimensions are $6,0 \mathrm{~m}$ $\mathrm{x} 2,4 \mathrm{~m}$, the edge $\varphi=-\Phi$ dimensions are $1,2 \mathrm{mx} 2,4 \mathrm{~m}$. The sheets are connected to each other by a soft metallic strip. The layout of the receiving and transmitting antenna arrangement is given in Fig. 2.

The distance between the receiving and transmitting antennas equals approximately $7,2 \mathrm{~m}$. Calculations have shown that the edge wave contribution $P \cdot C\left(\rho, \rho_{0}, \varphi, \varphi_{o}\right)$ into the common field $\Pi$ on the given distance, in the angle sector near the edge $\varphi=-\Phi$ was negligibly small. The experimental and calculation data are normalized relatively to the field intensity in the maximum of the first interference lobe of the transmitting antenna directivity pattern over the plane $(2 \Phi=\pi)$. In Fig. 3 along the abscissa axis the angle $\theta$ is depicted between the straight line from the point being the projection of the transmitting antenna onto the edge $\varphi=+\Phi$ to the observation point and the plane $\varphi=\Phi-\pi$. Besides data in the geometric optics approximation are presented by broken graphs. The comparison of both calculated by asymptotic formulas results and the experimental ones shows their good compliance.


Fig. 3. Amplitude dissipation patterns of the electric dipole field parallel to the wedge edge, when $\varphi_{0}=85^{\circ} 33 \prime 9.7 "$ and $\varphi_{0}=83^{\circ} 36^{\prime} 35.5 " \mathrm{~cm}\left(\lambda=3,05 \mathrm{~cm}, 2 \Phi=175^{\circ}, r_{w}=600 \mathrm{~cm}, R=720 \mathrm{~cm}\right)$

## 5. Results

In region D we have four boundaries: $\varphi_{1}=-\pi+\varphi_{0}, \varphi_{2}=-\pi+2 \Phi-\varphi_{0}, \varphi_{3}=\pi-2 \Phi-\varphi_{0}, \varphi_{4}=\pi-4 \Phi+\varphi_{0}$,
Two of them are situated in the region $-\Phi<\varphi<\Phi$. Two other are situated outside the wedge-shaped space. The roles of the boundaries $\varphi_{n}$ change with the source angular position adjustment to $\varphi_{0}$ or the wedge opening angle to $2 \Phi$ (with situation change). One of the boundaries goes outside the wedge limits $-\Phi<\varphi<\Phi$, while another one placed symmetrically relative to the edge $\varphi=+\Phi)$ appears in the region $-\Phi<\varphi<\Phi$. The field beam structure in the wedgeshaped region in the geometric optics approximation for each of four possible situations is presented on Fig. 4.

The field in the geometrical optics approximation suffers a break at the indicated boundaries ("light-shadow" boundary). The function $M\left(\varphi-\varphi_{n}\right)$ with the corresponding argument takes meaning equal to the half amplitude of the wave suffering the break. So the diffraction field has the transitional character at "light - shadow" boundaries of the propagation regions of the falling and reflected spherical waves. In the vicinity of the indicated boundaries energy redistribution takes place.


Fig. 4. Geometric optics waves ways

## 6. Conclusion

Asymptotic expansions (with $k \rightarrow \infty$ ) of Helmholtz equation solutions with the right part corresponding to the electric dipole were received. The dissipation field of the spherical wave at the wedge with perfectly conducting faces, with the wedge opening angle close to the straight angle is described by the sum of the fields of four half-shadow waves calculated with the help of Macdonald integral and the component describing the edge wave field. In the considered case of the given work it is possible to neglect the last component. The comparison of the calculated by asymptotic formulas results and the experimental ones shows their good compliance.

## 7. Acknowledgments

The work is fulfilled with the financial support of the Ministry of education and science of the Russian Federation.

## 8. References

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