Infinite Integrals With the Weighted Averages Algorithm

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Abstract

A new version of the weighted averages (WA) algorithm, called generalized WA, is introduced. Generalized WA exhibit a more compact formulation, devoid of iterative and recursive steps, and a wider range of applications. It is more robust, as it provides a unique formulation, valid for real and imaginary parameters. The implementation of the new version is easier and more economical in terms of basic operations. Preliminary numerical examples show the promise of the generalized WA that become the most interesting version in terms of simplicity, robustness and accuracy among the generic class of WA algorithms, which are currently recognized as the most competitive algorithms to evaluate Sommerfeld integral tails.

1 Introduction

The weighted averages (WA) algorithm was introduced within the microwave and antenna communities in the early eighties [1, 2], as a technique to evaluate the tail of the Sommerfeld integrals arising in the formulation of Green's functions for planar multilayered problems [3, 4]. The WA algorithm transforms these infinite integrals into an infinite sequence of partial finite integrals and acts upon this sequence as a convergence accelerator. Its original formulation was essentially heuristic and based on intuitive considerations. No attempt was made to connect it to the existing mathematical knowledge and its use was solely justified a posteriori by the excellent results it usually yielded. Then, in 1998, a seminal paper by Michalski [5] provided a rigorous frame for the WA, classifying it in the family of "integration-then-summation" procedures and identifying it mathematically as an extension of the Euler transformation [6, pp. 230]. The paper also stressed the fact, already mentioned in the original publications [1,2], that WA could also be efficiently applied to improper integrals, exhibiting oscillating divergent integrands and not being defined in the usual Riemann's sense [6, pp. 7]. After an intensive and thorough comparison with other algorithms, Michalski concluded that "*the W-transformation and the weighted averages method emerge as the most versatile and efficient currently known convergence accelerators for Sommerfeld integral tail*". This paper revisits the weighted averages algorithm and offers some new insights into it. A new, original version of WA is developed, based on a more general integral and hence exhibiting a wider range of applications. The resulting algorithm successfully compares with existing versions in terms of simplicity, computational speed and accuracy.

2 The Basic Integral

Let us start by considering a generic complex integral:

$$I(\gamma) = \int_a^\infty f(x) \exp(-\gamma x)\, dx \tag{1}$$

where

- $[a, \infty]$ is a semi-infinite interval on the real axis x,
- the function f will be considered, for the sake of simplicity, as being real-valued and behaving asymptotically as a power function $O(x^q)$; complex functions can be dealt with by considering successively their real and imaginary parts,
- γ is a complex parameter $\gamma = \alpha + j\beta$ satisfying the conditions: $\alpha \geq 0$ and $|\gamma| \neq 0$.

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If $\alpha > 0$, then the integral I in (1) is defined in the traditional Riemann sense, because the integrand converges exponentially at infinity. However, if $\alpha = 0$ and hence $\gamma = j\beta$, there is no exponential decrease to guarantee convergence. Indeed, if the asymptotic behavior of the function f is given by a power $q \geq 0$, the integrand doesn't converge and the integral is not defined in the Riemann sense. In these cases, a physical meaning can still be assigned to the integral I defining it in the Abel sense as

$$I(0 + j\beta) = \lim_{\alpha \to 0} I(\alpha + j\beta). \tag{2}$$

The Abel definition of convergence for improper integrals is a concept derived from the definition of "Abel's summability" for infinite divergent series and has a fully rigorous mathematical interpretation [7, pp.71]. However, any electrical engineer can understand "Abel's summability" in a very intuitive way, relating it to our understanding of electromagnetic phenomena arising in lossless media. A lossless medium is an ideal abstraction, exhibiting a pure imaginary propagation constant $\gamma = j\beta$. Its direct mathematical treatment involves frequently some difficulties. But the lossless situation can be always viewed as the limiting case of a physical lossy medium, with a complex propagation constant $\gamma = \alpha + j\beta$, when the losses vanish ($\alpha \to 0$).

3 The Generalized WA Procedure

Previously proposed WA algorithms are essentially more powerful versions of the Euler transformation [5], in which simple arithmetic means are replaced by weighted means. But it is well known that the Euler transformation can be easily generalized as Hölder means [7], acting simultaneously on the N members of a given sequence, rather than on two consecutive elements every time. The same strategy could be applied to WA. Hence, the generalized WA procedure will also start with a sequence of N partial integrals elements $I_n = I_n^{(0)}$. But now, the best possible evaluation of the infinite integral I will be obtained by performing a unique weighted average applied simultaneously to all the partial integrals.

To develop this generalized WA algorithm, let's remember the basic steps in the classic WA. The basic idea is to consider "partial" or "finite" integrals I_n, where the infinite integration interval $[a, \infty]$ is replaced by $[a, x_n]$, and the corresponding "remainders" R_n over the interval $[x_n, \infty]$:

$$I_n = \int_a^{x_n} f(x) \exp(-\gamma x)\, dx \tag{3}$$

$$R_n = I - I_n = \int_{x_n}^{\infty} f(x) \exp(-\gamma x)\, dx. \tag{4}$$

In the original WA formulation, the tails are approximated integrating by parts the integral (4). This yields an infinite series expansion for the remainder [3]:

$$R_n = I - I_n = \exp(-\gamma x_n) \sum_{k=1}^{\infty} f_n^{(k-1)} \gamma^{-k} \tag{5}$$

where the coefficients $f_n^{(k-1)}$ stand for $(k-1)$-th order derivative:

$$f_n^{(k)} = \left.\frac{\partial^k f(x)}{\partial x}\right|_{x=x_n}; \quad f_n^{(0)} = f(x_n). \tag{6}$$

We start writing the asymptotic expansion (5) as

$$-I \exp(\gamma x_n) + \sum_{k=1}^{N-1} f_n^{(k-1)} \gamma^{-k} = -I_n \exp(\gamma x_n) + O(\gamma^{-N}) \tag{7}$$

and applying it to N different integration limits x_n. Then, we can neglect the higher-order terms $O(\gamma^{-N})$, consider the resulting equations as a linear system and formally solve it for the unknown I. The determinant of such a linear system is

$$\begin{vmatrix} -\exp(\gamma x_1) & f_1^{(0)} & f_1^{(1)} & \cdots & f_1^{(N-2)} \\ -\exp(\gamma x_2) & f_2^{(0)} & f_2^{(1)} & \cdots & f_2^{(N-2)} \\ \vdots & \vdots & \vdots & & \vdots \\ -\exp(\gamma x_N) & f_N^{(0)} & f_N^{(1)} & \cdots & f_N^{(N-2)} \end{vmatrix}. \tag{8}$$

Hence, applying again the Cramer's rule, we can obtain our "best" estimation I_N^* of the infinite integral I as a quotient of determinants. Here, the denominator in this quotient is the determinant D_N and the numerator a determinant obtained replacing in the first column of (8) the elements $-\exp(\gamma x_n)$ by $-\exp(\gamma x_n) I_n$.

Both determinants can be expanded by their first column and the final expression is the *generalized WA formula*:

$$I_N^* = \frac{\sum\limits_{n=1}^{N} w_n I_n}{\sum\limits_{n=1}^{N} w_n}. \tag{9}$$

This is the sought-after linear combination of partial integrals allowing an estimation of the infinite integral which is optimum in the sense of the asymptotic development (7). In the above expression, the generalized weights are given by

$$w_n = (-1)^{n+1} \exp(\gamma x_n) M_n \tag{10}$$

where M_n are the minors of the determinant D_N, obtained by deleting the first column and the n-th row.

4 The Generalized Weights

The expression (10) for the generalized weights is of little practical interest, as it involves the computation of determinants whose elements are the values of the function $f(x)$ and its $N-2$ derivatives at N points x_n. Computing determinants is usually a cumbersome and time-consuming task. Fortunately enough, an interesting analytical treatment is possible in some cases of interest. It has been stated at the beginning of the paper that the function $f(x)$ was assumed to behave asymptotically as a power. In other words:

$$\lim_{x\to\infty} [f(x) - Cx^q] = 0 \tag{11}$$

with C and q being some real constants. If we replace in the determinant D_N the function $f(x)$ by its asymptotic approximation (11), we obtain for the minors M_n in (10), the value

$$M_n = C^{N-1} \left[\prod_{j=1}^{N-2} \frac{q!}{(q-j)!} \right] \left[\frac{1}{x_n} \prod_{j=1}^{N} x_j \right]^{q-N+2} V_n \tag{12}$$

where V_n are Vandermonde's determinants: $V_n \equiv V(1, x_1, x_2, \ldots, x_{n-1}, x_{n+1}, \ldots, x_N)$. These Vandermonde determinants have well known analytical expressions [8]. When used, the following expression for the weights, devoid of determinants, is obtained:

$$w_n = \frac{(-1)^{n+1} \exp(\gamma x_n)}{x_n^{q-N+2}} \prod_{\substack{j=1 \\ j\neq i}}^{N} |x_j - x_n|. \tag{13}$$

The above expression for the weights is perhaps not yet very tractable but it has the merit to be quite general. In particular, the abscissas x_n can have arbitrary values.

Equidistant abscissas and half-periods

Much simpler expressions for the weights are obtained if some restrictions are applied to the choice of the abscissas. If their values are not conditioned by external circumstances, the obvious choice is to select equidistant abscissas such as $x_{n+1} - x_n = h$. Then, the solution of the Vandermonde's determinant is even easier and a much more compact expression for the weights is obtained:

$$w_n = (-1)^{n+1} \exp\left(\gamma x_n\right) \binom{N-1}{n-1} x_n^{N-2-q}. \tag{14}$$

A further simplification is obtained if the abcissas are separated by half-periods. This was also the usual choice in the classic WA: $\beta(x_{n+1} - x_n) = \beta h = \pi$. This choice yields the final, simple expression:

$$w_n = \exp\left(\alpha x_n\right) \binom{N-1}{n-1} x_n^{N-2-q}. \tag{15}$$

This is a very convenient expression for a practical implementation, since the combinatorial numbers can be easily computed by recursion. Moreover, the weights are always real and positive and this makes of expression (9) a true weighted means of the partial integrals.

5 Conclusion

The new generalized version of the WA algorithm shows clear advantages from the points of view of theory, implementation and accuracy. Its theoretical construction is straightforward and fully supported by a rigorous mathematical background, being a generalization of the well established Hölder means used to sum divergent series. The iterative and recursive nature of classic WA is eliminated and replaced by a unique weighted means, where the weights are defined in a univocal way. This results in a reduction of the number of operations. In addition, it is expected that the new WA should be less sensitive to the propagation of round-off errors. Due to the simplicity of its formulation, the implemented algorithm is necessarily simpler than a classic WA implementation. Finally, the generalized weighted averages algorithm provides a robust approach since the same formula is valid for all complex values of the parameter $\gamma = \alpha + j\beta$, ranging from pure real (but no negative!) to pure imaginary values, while the accuracy is well maintained all over the range. Extensive results justifying and developing the above will be presented during the conference.

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