# Relativistic Invariance of the Time-Domain Waveguide Modes

*Oleg A. Tretyakov*[1], and *Ozlem Akgun*[2]

[1]Gebze Institute of Technology, 41400, Cayirova, Gebze, Kocaeli, Turkey; o.tretyakov@gmail.com
[2]Gebze Institute of Technology, 41400, Cayirova, Gebze, Kocaeli, Turkey; oakgun@gyte.edu.tr

**Abstract**

The relativistic invariance of a complete set of time-domain modes under Lorentz transformation is proved. The time-domain modes are exhibited as the particular solutions to the system of Maxwell's equations with time derivative. The waveguide surface has the properties of perfect electric conductor. The modal fields are presented via transverse-longitudinal decompositions. Every field component is a product of a modal amplitude depending on longitudinal coordinate and time, and an element of modal basis in the waveguide cross section. The modal basis is specified in a general form. Exact explicit solutions for the modal amplitudes are presented.

## 1 Introduction: Maxwell's Equations and Lorentz Transformation

In this section, the formulation of problem (1) which results, ultimately, in a complete set of the time-domain modes as the particular solutions is given. Besides, the relevant information on the Lorentz transformation in electrodynamics (needed for analysis of relativistic properties of the modes) is listed.

A medium-free waveguide has a cross-section domain, $S$, bounded by a closed singly connected contour, $L$. None of possible *inner* angles of $L$ (i.e., measured *within* $S$) exceed $\pi$. The cross-section domain, $S$, maintains its size and form along the waveguide axis, $Oz$. The waveguide surface has the properties of perfect electric conductor. A right-handed triplet of the mutually orthogonal unit vectors $(\mathbf{z},\mathbf{l},\mathbf{n})$ is introduced as $\mathbf{z}\times\mathbf{l}=\mathbf{n}$, and so on. The vector $\mathbf{z}$ is oriented along the axis $Oz$, $\mathbf{l}$ is tangential to the contour $L$, $\mathbf{n}$ is the outward normal to the domain $S$. A boundary-value time-domain problem is stated in a common way as

$$\nabla\times\mathcal{H}=\epsilon_0\partial_t\mathcal{E},\quad \nabla\times\mathcal{E}=-\mu_0\partial_t\mathcal{H}\,;\qquad (\mathbf{n}\cdot\mathcal{H})\,|_L=0,\quad (\mathbf{l}\cdot\mathcal{E})\,|_L=0,\quad (\mathbf{z}\cdot\mathcal{E})\,|_L=0 \tag{1}$$

where $\mathcal{E}\equiv\mathcal{E}\,(\mathbf{R},t)$ and $\mathcal{H}\equiv\mathcal{H}\,(\mathbf{R},t)$ are the electric and magnetic strength vectors, respectively, $\mathbf{R}$ is a three-component position vector of a point of observation within the waveguide, $t$ is an observation time. Introduce so-called transverse-longitudinal decompositions of the three-component field vectors and $\nabla$ as

$$\mathcal{E}\,(\mathbf{R},t)=\mathbf{E}\,(\mathbf{r},z,t)+\mathbf{z}\,E_z\,(\mathbf{r},z,t)\,,\quad \mathcal{H}\,(\mathbf{R},t)=\mathbf{H}\,(\mathbf{r},z,t)+\mathbf{z}\,H_z\,(\mathbf{r},z,t)\,;\quad \mathbf{R}=\mathbf{r}+\mathbf{z}\,z,\ \nabla=\nabla_\perp+\mathbf{z}\,\partial_z \tag{2}$$

where $\mathbf{r}$ is the two-component projection on the domain $S$ of the vector $\mathbf{R}$, operator $\nabla_\perp$ acts on the transverse coordinates, $\mathbf{r}$, only, $\mathbf{E}$ and $\mathbf{H}$ are the two-component projections on the domain $S$ of the vectors $\mathcal{E}$ and $\mathcal{H}$.

Associate the set of coordinates and time $\{\mathbf{r},z,t\}$ with an inertial reference frame $\mathbf{F}$ which is in a state of rest under definition. Introduce a new inertial reference frame, say $\bar{\mathbf{F}}$, specified by a set $\{\bar{\mathbf{r}},\bar{z},\bar{t}\}$, which is in translational motion with a constant velocity $\mathbf{v}=\mathbf{z}\,v$ *along* the axis $Oz$. Correspondence between $\{\mathbf{r},z,t\}$ and $\{\bar{\mathbf{r}},\bar{z},\bar{t}\}$ and vice-versa are specified by the *direct* and *inverse* ($v\to -v$) Lorentz transformation as

$$\underline{\textit{direct}}\text{:}\ \bar{\mathbf{r}}=\mathbf{r},\quad \bar{z}=\gamma\,(z-vt)\,,\quad \bar{t}=\gamma\left(t-z\,v/c^2\right);\ \ \underline{\textit{inverse}}\text{:}\ \mathbf{r}=\bar{\mathbf{r}},\quad z=\gamma\,(\bar{z}+v\,\bar{t})\,,\quad t=\gamma\left(\bar{t}+\bar{z}\,v/c^2\right) \tag{3}$$

where $\gamma=1/\sqrt{1-\beta^2}$, $\beta=v/c$, see [1]. Evidently, operator $\nabla_\perp$ is invariant in $\mathbf{F}$ and $\bar{\mathbf{F}}$. Similar to the field decompositions (2) in $\mathbf{F}$, *the same* field in $\bar{\mathbf{F}}$ is presentable as

$$\mathcal{E}\left(\bar{\mathbf{R}},\bar{t}\right)=\bar{\mathbf{E}}\,(\bar{\mathbf{r}},\bar{z},\bar{t})+\mathbf{z}\,\bar{E}_z\,(\bar{\mathbf{r}},\bar{z},\bar{t})\quad\text{and}\quad \mathcal{H}\left(\bar{\mathbf{R}},\bar{t}\right)=\bar{\mathbf{H}}\,(\bar{\mathbf{r}},\bar{z},\bar{t})+\mathbf{z}\,\bar{H}_z\,(\bar{\mathbf{r}},\bar{z},\bar{t})\,. \tag{4}$$

The well-known relationships (see [1], for example) between the field components in (2) and (4) hold as

$$\begin{aligned}\bar{E}_z\,(\bar{\mathbf{r}},\bar{z},\bar{t})&=E_z\,(\mathbf{r},z,t)\,,\qquad \bar{\mathbf{E}}\,(\bar{\mathbf{r}},\bar{z},\bar{t})=\gamma\left[\mathbf{E}\,(\mathbf{r},z,t)+v\,\mathbf{z}\times\mu_0\mathbf{H}\,(\mathbf{r},z,t)\right],\\ \bar{H}_z\,(\bar{\mathbf{r}},\bar{z},\bar{t})&=H_z\,(\mathbf{r},z,t)\,,\qquad \bar{\mathbf{H}}\,(\bar{\mathbf{r}},\bar{z},\bar{t})=\gamma\left[\mathbf{H}\,(\mathbf{r},z,t)-\left(v/c^2\right)\,\mathbf{z}\times\mu_0^{-1}\mathbf{E}\,(\mathbf{r},z,t)\right].\end{aligned} \tag{5}$$

978-1-4244-6051-9/11/$26.00 ©2011 IEEE

## 2 Relativistic Transverse-Electric $(TE)$ Time-Domain Waveguide Modes

We analyze relativistic properties of the complete sets of $TE$ and $TM$ time-domain waveguide modes derived in [2,3]. Herein, we slightly change notation (adopted earlier) in order to extract explicitly the physical dimensions of electric and magnetic quantities which are necessary in practical calculations.

The derivation of $TE-$modes starts with solving Neumann boundary eigenvalue problem for transverse Laplacian, $\nabla_\perp^2$, the first pair of equations in (6). The last equation for $\mathcal{N}_n^{TE}$ involves N (*newton*), physical dimension of force. This results in the normalization constant, $\mathcal{N}_n^{TE}$, as a *dimensional* one. Thus,

$$\nabla_\perp^2\,\psi_n(\mathbf{r}) + \nu_n^2\,\psi_n(\mathbf{r}) = 0\,, \quad \mathbf{n}\cdot\nabla_\perp\psi_n(\mathbf{r})|_L = 0; \qquad \tfrac{\nu_n^2}{S}\int_S \left|\mathcal{N}_n^{TE}\psi_n(\mathbf{r})\right|^2 ds = 1\,\mathrm{N} \tag{6}$$

where $\mathbf{r}\in S$, the eigenvalues $\nu_n^2 > 0$ $(n = 1, 2, ...)$ are discrete values of a real-valued spectral parameter $\nu$, subscript $(_n)$ regulates the position of eigenvalues on a real axis in increasing order of their numerical values, $\psi_n$ $(n = 1, 2, ...)$ are the eigensolutions corresponding to these eigenvalues.

Problem (1) yields an infinite set of particular solutions for the transverse-electric fields composed as

$$\begin{aligned}\mathcal{E}_n^{TE}(\mathbf{r},z,t) &= \mathcal{A}_n^h(z,t)\,[\epsilon_0^{-\frac{1}{2}}\mathcal{N}_n^{TE}\,\nu_n\nabla_\perp\psi_n(\mathbf{r})\times\mathbf{z}]\\ \mathcal{H}_n^{TE}(\mathbf{r},z,t) &= \mathcal{B}_n^h(z,t)\,[\mu_0^{-\frac{1}{2}}\mathcal{N}_n^{TE}\,\nu_n\nabla_\perp\psi_n(\mathbf{r})] + \mathbf{z}\,h_n(z,t)\,[\mu_0^{-\frac{1}{2}}\mathcal{N}_n^{TE}\,\nu_n^2\psi_n(\mathbf{r})]\end{aligned} \tag{7}$$

where the scalar coefficients, $\mathcal{A}_n^h$, $\mathcal{B}_n^h$ and $h_n$, have physical sense of the modal amplitudes (dimensionless quantities). The bracketed factors originate a dimensional *modal basis* in the waveguide cross section. The basis elements of $\mathcal{E}_n^{TE}-$fields have physical dimension of $\mathrm{V\,m^{-1}}$ (volt per meter) and the basis elements of $\mathcal{H}_n^{TE}-$fields have dimension of $\mathrm{A\,m^{-1}}$ (ampere per meter). Problem (6) has one more solution, $\psi_0(\mathbf{r})$, corresponding to $\nu_0^2$=0, which generates a static field, $\mathcal{E}_0^{TE}=\mathbf{0}$ and $\mathcal{H}_0^{TE}=\mathbf{z}\,C\,\mathrm{A\,m^{-1}}$, where $C$ is a constant.

**Remark 1** *It can be verified that the constant $\mathcal{N}_n^{TE}$ is proportional to $\nu_n^{-1}\,\mathrm{N}^{\frac{1}{2}}$, quantity $\nu_n$ has dimension of $\mathrm{m}^{-1}$ (inverse meter), quantity $\epsilon_0^{-\frac{1}{2}}\,\mathrm{N}^{\frac{1}{2}}$ has dimension of* V *(volt) and quantity $\mu_0^{-\frac{1}{2}}\,\mathrm{N}^{\frac{1}{2}}$ has dimension of* A *(ampere), operator $\nabla_\perp$ has physical dimension of $\mathrm{m}^{-1}$.*

As it follows from (3), the transverse waveguide coordinates (and hence, the solutions to Neumann problem) are *universal* in the relativistic sense. Specifically, the modal basis in (7) is valid both for $\mathbf{F}$ and $\bar{\mathbf{F}}$. It is known from time-harmonic field theory that the eigenvalues, $\nu_n^2$, from Neumann problem specify the *cut-off* frequencies of the $TE-$modes, $\omega_n^{TE}$, as follows: $\omega_n^{TE} = c\nu_n = 2\pi/T_n^{TE}$. Respectively, $\nu_n = 2\pi/\lambda_n^{TE}$ where $\lambda_n^{TE}$ is the cut-off wavelength. Hence, these waveguide characteristics, $\lambda_n^{TE}$ and $T_n^{TE}$, are invariant both in $\mathbf{F}$ and $\bar{\mathbf{F}}$. This suggests to introduce *dimensionless* variables as follows

$$\xi = \nu_n\,z = 2\pi\,z\,/\,\lambda_n^{TE} \quad \text{and} \quad \tau = \nu_n c\,t = 2\pi\,t\,/\,T_n^{TE} \tag{8}$$

and replace the argument $(z,t)$ of the modal amplitudes with $(\xi,\tau)$ in (7). The modal amplitude of longitudinal component of *magnetic* field, $h_n(\xi,\tau)$, satisfies the Klein-Gordon equation, which is the first one in (9), and serves as a potential for the other two modal amplitudes as

$$\partial_\tau^2 h_n(\xi,\tau) - \partial_\xi^2 h_n(\xi,\tau) + h_n(\xi,\tau) = 0 \quad \text{and} \quad \mathcal{A}_n^{TE}(\xi,\tau) = -\partial_\tau h_n(\xi,\tau)\,, \quad \mathcal{B}_n^{TE}(\xi,\tau) = \partial_\xi h_n(\xi,\tau)\,, \tag{9}$$

see [3]. The differential procedures, $\partial_\xi$ and $\partial_\tau$, in $\mathbf{F}$ have their equivalents in $\bar{\mathbf{F}}$ as

$$\partial_\xi = \gamma\left(\partial_{\bar\xi} - \beta\,\partial_{\bar\tau}\right), \quad \partial_\tau = \gamma\left(\partial_{\bar\tau} - \beta\,\partial_{\bar\xi}\right) \quad \text{where} \quad \bar\xi = \nu_n\,\bar z, \quad \bar\tau = c\nu_n\,\bar t\,. \tag{10}$$

Take now the modal field (7), given in $\mathbf{F}$, and find out its equivalent in $\bar{\mathbf{F}}$ with using the Lorentz transformation (5) for the transverse field components. Careful analysis shows that the modal field has the same form in $\bar{\mathbf{F}}$ as in $\mathbf{F}$. In other words, it is enough to take the field (7) with the set of independent variables $\{\mathbf{r},z,t\}$ and replace it with another set, $\{\bar{\mathbf{r}},\bar z,\bar t\}$. The velocity $\mathbf{v}$ (a characteristics of $\bar{\mathbf{F}}$) does not appear in the fields in $\bar{\mathbf{F}}$. Thus, the time-domain field (7) is invariant under the Lorentz transformation.

## 3 Relativistic Transverse-Magnetic $(TM)$ Time-Domain Waveguide Modes

The set of $TM$ time-domain waveguide modes was derived, as well, in [2,3]. Herein, we shall give the $TM-$modal fields in new notation which provide the electric and magnetic strength vectors with the dimensions of $\mathrm{V\,m^{-1}}$ and $\mathrm{A\,m^{-1}}$ in SI units, respectively. Relativistic analysis of the fields is given briefly.

The derivation of $TM-$modes starts with solving the Dirichlet boundary eigenvalue problem for transverse Laplacian, $\nabla_\perp^2$, as the first pair of equations in (11). The last equation herein involves the normalization constant, $\mathcal{N}_n^{TM}$, as a dimensional quantity. Thus,

$$\nabla_\perp^2\,\phi_n(\mathbf{r}) + \varkappa_n^2\,\phi_n(\mathbf{r}) = 0\,, \quad \phi_n(\mathbf{r})|_L = 0\,; \qquad \tfrac{\varkappa_n^2}{S}\int_S \left|\mathcal{N}_n^{TM}\phi_n(\mathbf{r})\right|^2 ds = 1\,\mathrm{N} \tag{11}$$

where N is the dimension of force, $\mathbf{r}\in S$, the eigenvalues $\varkappa_n^2 > 0$ $(n = 1, 2, ...)$ are discrete values of a real-valued spectral parameter $\varkappa$, subscript $(_n)$ regulates the position of eigenvalues on a real axis in increasing order of their numerical values, $\phi_n$ $(n = 1, 2, ...)$ are the eigensolutions corresponding to these eigenvalues.

The set of potential $\{\phi_n(\mathbf{r})\}_{n=1}^{\infty}$ generates an infinite set of particular solutions to problem (1) as

$$\begin{aligned}
&\mathcal{H}_n^{TM}(\mathbf{r}, z, t) = \mathcal{A}_n^e(z, t)\,[\mathbf{z}\times\mu_0^{-\frac{1}{2}}\mathcal{N}_n^{TM}\varkappa_n\nabla_\perp\phi_n(\mathbf{r})]\\
&\mathcal{E}_n^{TM}(\mathbf{r}, z, t) = \mathcal{B}_n^e(z, t)\,[\epsilon_0^{-\frac{1}{2}}\mathcal{N}_n^{TM}\,\varkappa_n\nabla_\perp\phi_n(\mathbf{r})] + \mathbf{z}\,e_n(z, t)\,[\epsilon_0^{-\frac{1}{2}}\mathcal{N}_n^{TM}\,\varkappa_n^2\phi_n(\mathbf{r})]
\end{aligned} \tag{12}$$

where the bracketed factors are the elements of modal basis in the waveguide cross section, $S$. We may assume that they are already known (along with their physical dimensions) since the Dirichlet problem is well studied in mathematics and in numerous applications. The modal amplitudes, $\mathcal{A}_n^e$, $\mathcal{B}_n^e$, and $e_n$, in (12) are dimensionless factors. It is convenient to replace the variables $(z, t)$ with dimensionless ones, $(\xi, \tau)$, as

$$\xi = \varkappa_n\,z = 2\pi\,z\,/\,\lambda_n^{TM} \quad \text{and} \quad \tau = \varkappa_n c\,t = 2\pi\,t\,/\,T_n^{TM} \tag{13}$$

where $\lambda_n^{TM}$ is the cut-off wavelength of $TM$ time-harmonic modes and $T_n^{TM} = \lambda_n^{TM}/c$, $c$ is the velocity of light. The modal amplitude of longitudinal component of *electric* field, $e_n(\xi, \tau)$, is the solution to Klein-Gordon equation that generates the other two modal amplitudes as follows

$$\partial_\tau^2 e_n(\xi, \tau) - \partial_\xi^2 e_n(\xi, \tau) + e_n(\xi, \tau) = 0 \quad \text{and} \quad \mathcal{A}_n^e(\xi, \tau) = -\partial_\tau e_n(\xi, \tau)\,, \quad \mathcal{B}_n^e(\xi, \tau) = \partial_\xi e_n(\xi, \tau)\,. \tag{14}$$

Manipulations with the Lorentz transformation (5) for transverse field components in (12) (and making use of (10) in passing) validate that the fields (12) possess all the relativistic properties.

The two modal amplitude problems, (9) and (14), can be rewritten universally as

$$\partial_\tau^2\mathfrak{f}(\xi, \tau) - \partial_\xi^2\mathfrak{f}(\xi, \tau) + \mathfrak{f}(\xi, \tau) = 0 \quad \text{and} \quad \mathcal{A}(\xi, \tau) = -\partial_\tau\mathfrak{f}(\xi, \tau)\,, \quad \mathcal{B}(\xi, \tau) = \partial_\xi\mathfrak{f}(\xi, \tau)\,. \tag{15}$$

When the variables $(\xi, \tau)$ are chosen by (13), then $\mathfrak{f}(\xi, \tau)$ is $e_n(\xi, \tau)$ and $\mathcal{A} \equiv \mathcal{A}_n^e$ and $\mathcal{B} \equiv \mathcal{B}_n^e$. When the variables $(\xi, \tau)$ are chosen by (8), then $\mathfrak{f}(\xi, \tau)$ is $h_n(\xi, \tau)$ and $\mathcal{A} \equiv \mathcal{A}_n^h$ and $\mathcal{B} \equiv \mathcal{B}_n^h$. In practice, this observation gives the possibility to study the $TE$ and $TM$ time-domain waveguide modes *in parallel.* It turns out that Klein-Gordon equation has remarkable properties resulting in explicit solutions for the modal amplitudes. They were disclosed within the scope of group theory and have sense of substitutions of new variables to the Klein-Gordon equation. Complete list of *eleven* possible substitutions is given in [3].

Possible exact explicit solutions to the problem (15) have the following form:

$$\begin{aligned}
&\alpha > 0: \quad \mathfrak{f}_\alpha(\xi, \tau) = \left(\tfrac{\tau-|\xi|}{\tau+|\xi|}\right)^{\alpha/2} J_\alpha\left(\sqrt{\tau^2 - |\xi|^2}\right), \quad \mathcal{A}_\alpha = -\tfrac{1}{2}\left(\mathfrak{f}_{\alpha-1} - \mathfrak{f}_{\alpha+1}\right), \quad \mathcal{B}_\alpha = -\tfrac{1}{2}\left(\mathfrak{f}_{\alpha-1} + \mathfrak{f}_{\alpha+1}\right)\\
&\alpha = 0: \quad \mathfrak{f}_0(\xi, \tau) = J_0\left(\sqrt{\tau^2 - |\xi|^2}\right), \qquad \mathcal{A}_0 = \tau\,\frac{J_1\left(\sqrt{\tau^2-|\xi|^2}\right)}{\sqrt{\tau^2-|\xi|^2}}, \qquad \mathcal{B}_0 = \xi\,\frac{J_1\left(\sqrt{\tau^2-|\xi|^2}\right)}{\sqrt{\tau^2-|\xi|^2}}
\end{aligned} \tag{16}$$

where $\alpha$ is a free numerical parameter, $J_\alpha(\cdot)$ is the Bessel function of order $\alpha$, and $\tau \geq |\xi|$. When $\alpha$ is an integer ($\alpha$=$m$=$0, 1, 2, \ldots$), then $J_m(\cdot)$ are the *cylindrical* Bessel functions. When $\alpha$ has a half-integer value as $\alpha$=$m$+$\frac{1}{2}$, then $J_{m+\frac{1}{2}}(\cdot)$ are the *spherical* Bessel functions. The Bessel functions, $J_a(\cdot)$, with $\alpha = \frac{1}{4}$ and $\frac{3}{4}$ are convertible to the *parabolic cylinder* functions. The Bessel functions, $J_\alpha(\cdot)$, with $\alpha$=1/3 and 2/3 are convertible to the *Airy* functions. One can assign even a complex value to the parameter, $\alpha$, when needed.

Suppose that a modal field has a *beginning in time* (say, at $\tau = 0$) in the waveguide cross section located at coordinate $\xi = 0$ and propagates along the axis $Oz$. Mathematically, we have to solve the Klein-Gordon equation supplemented with appropriate *"initial conditions"* as follows

$$\partial_\tau^2 \mathfrak{F}(\xi,\tau) - \partial_\xi^2 \mathfrak{F}(\xi,\tau) + \mathfrak{F}(\xi,\tau) = 0 \quad \text{and} \quad \mathfrak{F}(0,\tau) = \mathfrak{f}^0(\tau), \quad \partial_\tau \mathfrak{F}(0,\tau) = \mathcal{A}^0(\tau) \tag{17}$$

where $\mathfrak{f}^0(\tau) \equiv \mathfrak{f}(0,\tau)$ and $\mathcal{A}^0(\tau) \equiv \mathcal{A}(0,\tau)$ should be given. We have to read the solution to problem (17) in compliance with the *causality principle* as follows: 1) $\mathfrak{F}(\xi,\tau) \equiv 0$ if $\tau < 0$ since there is no modal signal *before* $\tau = 0$ at $\xi = 0$, 2) $\mathfrak{F}(\xi,\tau) \equiv 0$ if $\xi > \tau$ since the signal can not *overcome* the distance $\xi = \tau$ starting from $\xi = 0$ at instant $\tau = 0$ (in accordance with the second postulate of the special relativity theory), 3) within the interval $0 \leq \xi \leq \tau$, the solution is nontrivial, $\mathfrak{F}(\xi,\tau) \neq 0$, accordingly to the postulate.

## 4 Conclusion

The formulation of time-domain waveguide problem is given and the solution to that is proposed in the form of the transverse-longitudinal field decompositions. Every field component is a product of two factors: One, depending on the transverse coordinates, is a vector element of the modal basis in the waveguide cross section. The other is a scalar modal amplitude depending on the axial coordinate and time. The modal basis is universal for all the inertial reference frames moving with a constant velocity along the waveguide axis. The modal amplitudes of the longitudinal field components are invariant under the Lorentz transformation. The modal amplitudes of the transverse components suffer from relativistic transformations in such a way, however, the total field remain invariant under the Lorentz transformation. Exact explicit solutions for the relativistic modal amplitudes are presented. Graphical illustrations will be exhibited in the presentation.

## References


[1] J. G. Van Bladel, *Electromagnetic Fields, Second Edition,* John Wiley&Sons, Inc., Hoboken, New Jersey, 2007, pp. 947-953.

[2] O. A. Tretyakov, "Essentials of nonstationary and nonlinear electromagnetic field theory," in M. Hashimoto, M. Idemen and O. A. Tretyakov (eds.), *Analytical and Numerical Methods in Electromagnetic Wave Theory*, Tokyo, Science House Co., Ltd., 1993, pp. 123-146.

[3] O. A. Tretyakov, O. Akgun, "Derivation of Klein-Gordon Equation from Maxwell's Equations and Study of Relativistic Time-Domain Waveguide Modes, "Progress In Electromagnetics Research, vol. 105, pp. 171-191, June 2010.