A rigorous model for axially-symmetric radiators of pulsed waves

***Olena Shafalyuk*[1] *and Paul Smith*[2]**

[1]Department of Mathematics, Macquarie University, North Ryde, NSW, 2109, Australia,
e-mail: olena.shafalyuk@mq.edu.au

[2] Department of Mathematics, Macquarie University, e-mail: paul.smith@mq.edu.au

Abstract

In the present work, we solve a problem of the appropriate and efficient truncation of the computational domain (a principal problem of computational electrodynamics) in so-called open problems (the problems where the computational domain is infinite along one or more spatial coordinates) for the case of TE- and TM-waves in axially-symmetrical open compact resonators with waveguide feed lines. Also, a number of questions have been considered that occur when solving far-field problems and problems involving extended sources or sources located in the far-zone.

1. Introduction

Widespread adoption of microwave applications, which has continued at an accelerated pace in recent years, gives rise to new theoretical and engineering problems dealing with the design and modernization of various contemporary microwave devices and their units. Of particular interest is the extension of functional capabilities of easily engineered assemblages made of inexpensive materials with well-studied electrical characteristics. Among them are axially-symmetric metal-dielectric elements. They are easily mated; available technologies are sufficient to make laboratory patterns and to produce end items; open waveguide resonators (or waveguide units) employing inhomogeneities in circular and coaxial waveguides, as well as radiators of this kind, are capable of changing the parameters of the exciting monochromatic and pulsed TE- and TM- waves over wide limits. It is obvious that analysis and systematization of the phenomena occurring in axially-symmetric structures during propagation, radiation, and scattering of electromagnetic waves form the theoretical base for fresh and efficient engineering solutions.

The present work is concerned with developing a rigorous algorithm for solving initial boundary-value problems for axially-symmetric open resonators and radiators of pulsed TE- and TM- waves. We have constructed novel exact absorbing boundary conditions truncating the domain of computation in the finite-difference numerical schemes when solving the originally open model problems. We also develop the analytical approach for solving the problem of equivalent replacement of open initial boundary-value electrodynamic problems with the closed ones [1].

2. Formulation of the Problem

In Fig. 1, the cross-section of a model for an open axially-symmetrical ($\partial/\partial\phi \equiv 0$) resonant structure is shown, where $\{\rho,\phi,z\}$ are cylindrical and $\{\rho,\vartheta,\phi\}$ are spherical coordinates. By $\Sigma = \Sigma_\phi \times [0,2\pi]$ we denote perfectly conducting surfaces obtained by rotating the curve Σ_ϕ about the z -axis; $\Sigma^{\varepsilon,\sigma} = \Sigma_\phi^{\varepsilon,\sigma} \times [0,2\pi]$ is a similarly defined surface across which the relative permittivity $\varepsilon(g)$ and specific conductivity $\sigma_0(g) = \eta_0^{-1}\sigma(g)$ change step-wise; these quantities are piecewise constant inside Ω_{int} and take free space values outside. Here, $g = \{\rho, z\}$; $\eta_0 = (\mu_0/\varepsilon_0)^{1/2}$ is the impedance of free space; ε_0, and μ_0 are the electric and magnetic constants of vacuum.

The two-dimensional initial boundary-value problem describing the pulsed axially-symmetrical TE_{0n}- ($E_\rho = E_z = H_\phi \equiv 0$) and TM_{0n}- ($H_\rho = H_z = E_\phi \equiv 0$) wave distribution in open structures of this kind is given by

978-1-4244-6051-9/11/$26.00 ©2011 IEEE

$$
\begin{cases}
\left[-\varepsilon(g)\frac{\partial^2}{\partial t^2}-\sigma(g)\frac{\partial}{\partial t}+\frac{\partial^2}{\partial z^2}+\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\right)\right]U(g,t)=F(g,t), \quad t>0, \quad g\in\Omega \\
U(g,t)\big|_{t=0}=\varphi(g), \quad \frac{\partial}{\partial t}U(g,t)\Big|_{t=0}=\psi(g), \quad g=\{\rho,z\}\in\bar{\Omega} \\
E_{tg}(p,t)\big|_{p=\{\rho,\phi,z\}\in\Sigma}=0, \quad t\ge 0 \\
E_{tg}(p,t) \text{ and } H_{tg}(p,t) \text{ are continuous when crossing } \Sigma^{\varepsilon,\sigma} \\
U(0,z,t)=0, \quad |z|<\infty, \quad t\ge 0 \\
D_1\left[U(g,t)-U^{i(1)}(g,t)\right]\Big|_{g\in\Gamma_1}=0, \quad D_2\left[U(g,t)\right]\Big|_{g\in\Gamma_2}=0, \quad t\ge 0
\end{cases}, \qquad (1)
$$

where $\vec{E}=\{E_\rho,E_\varphi,E_z\}$ and $\vec{H}=\{H_\rho,H_\varphi,H_z\}$ are the electric and magnetic field vectors; $U(g,t)=E_\phi(g,t)$ for TE_{0n}-waves and $U(g,t)=H_\phi(g,t)$ for TM_{0n}-waves [1]. The SI system of units is used. The variable t which being the product of the real time by the velocity of light in free space has the dimension of length.

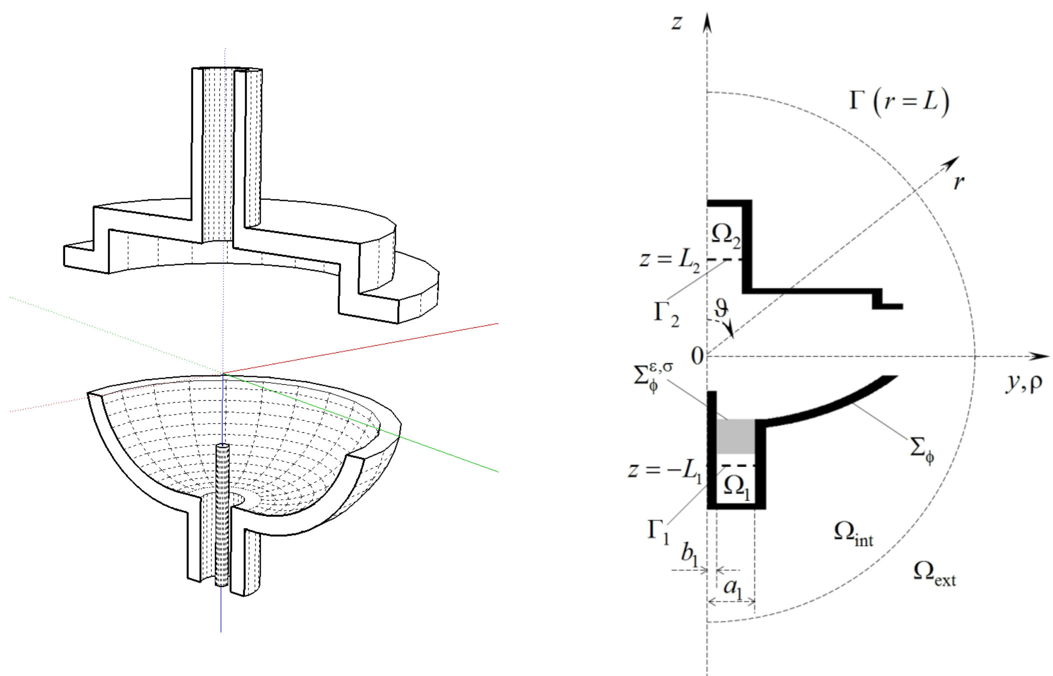

Fig. 1. Geometry of the problem.

The domain of analysis Ω is the part of the half-plane $\phi=\pi/2$ bounded by the contours Σ_ϕ together with the artificial boundaries Γ_j (input and output ports) in the virtual waveguides Ω_j, $j=1,2$. The regions $\Omega_{\text{int}}=\{g=\{r,\vartheta\}\in\Omega:\ r<L\}$ and Ω_{ext} (free space), such that $\Omega=\Omega_{\text{int}}\cup\Omega_{\text{ext}}\cup\Gamma$, are separated by the virtual boundary $\Gamma=\{g=\{r,\vartheta\}\in\Omega:\ r=L\}$.

The functions $F(g,t)$, $\varphi(g)$, $\psi(g)$, $\sigma(g)$, and $\varepsilon(g)-1$ which are finite in the closure $\bar{\Omega}$ of Ω are supposed to satisfy the hypotheses of the theorem on the unique solvability of problem (1) in the Sobolev space

$\mathbf{W}_2^1\left(\Omega^T\right)$, $\Omega^T=\Omega\times(0;T)$ where $T<\infty$ is the observation time [2]. The 'current' and 'instantaneous' sources given by the functions $F(g,t)$ and $\varphi(g)$, $\psi(g)$ as well as all scattering elements given by the functions $\varepsilon(g)$, $\sigma(g)$ and by the contours Σ_ϕ and $\Sigma_\phi^{\varepsilon,\sigma}$ are located in the region Ω_{int}. In axially-symmetrical problems, at points g such that $\rho=0$, only H_z or E_z fields components are nonzero. Hence it follows that $U(0,z,t)=0$; $|z|<\infty$, $t\geq 0$ in (1).

Equations $D_{1,2}=0$ in (1) give the exact absorbing conditions for the outgoing pulsed waves $U^{s(1)}(g,t)=U(g,t)-U^{i(1)}(g,t)$ and $U^{s(2)}(g,t)=U(g,t)$ traveling into the virtual waveguides Ω_1 and Ω_2, respectively [1]. $U^{i(1)}(g,t)$ is the pulsed wave that excites the axially-symmetrical structure from the circular or coaxial circular waveguide Ω_1. It is assumed that by the time $t=0$ this wave has not yet reached the boundary Γ_1.

By using these conditions, we simplify substantially the model simulating an actual electrodynamic structure: the Ω_j-domains are excluded from consideration while the operators D_j describe wave transformation on the boundaries Γ_j that separate regular feeding waveguides from the radiating unit. The operators D_j are constructed such that a wave incident on Γ_j from the region Ω_{int} passes into the virtual domain Ω_j as if into a regular waveguide – without deformations or reflections. In other words, it is absorbed completely by the boundary Γ_j. Therefore, we call the boundary conditions $D_{1,2}=0$ as well as the other conditions of this kind 'exact absorbing conditions'.

3. Basic Results

1. When constructing the exact absorbing condition for the wave $U(g,t)$ crossing the artificial spherical boundary Γ (Fig. 1), we followed the sequence of transformations widely used in the theory of hyperbolic equations – incomplete separation of variables in initial boundary-value problems for telegraph or wave equations, integral transformations in the problems for one-dimensional Klein-Gordon equations, solution of the auxiliary boundary-value problems for ordinary differential equations, and inverse integral transforms. As a result, we arrive at the following formula

$$\begin{aligned}
U(L,\vartheta,t)=\frac{1}{2}\sum_{n=1}^{\infty}\Bigg\{&\int_{t-2L}^{t}\Bigg[\left(\frac{t-\tau}{L\sqrt{4L^2-(t-\tau)^2}}P_n^1\left(1-\frac{(t-\tau)^2}{2L^2}\right)-\frac{1}{L}P_n\left(1-\frac{(t-\tau)^2}{2L^2}\right)\right)\times\\
&\times\int_0^{\pi}U(L,\vartheta_1,\tau)\,\tilde{\mu}_n(\cos\vartheta_1)\sin\vartheta_1 d\vartheta_1-\\
&-P_n\left(1-\frac{(t-\tau)^2}{2L^2}\right)\int_0^{\pi}\left.\frac{\partial U(r,\vartheta_1,\tau)}{\partial r}\right|_{r=L}\tilde{\mu}_n(\cos\vartheta_1)\sin\vartheta_1 d\vartheta_1\Bigg]d\tau+\\
&+\int_0^{\pi}\left[U(L,\vartheta_1,t)+(-1)^n U(L,\vartheta_1,t-2L)\right]\tilde{\mu}_n(\cos\vartheta_1)\sin\vartheta_1 d\vartheta_1\Bigg\}\tilde{\mu}_n(\cos\vartheta),\qquad 0\leq\vartheta\leq\pi,
\end{aligned}\tag{2}$$

which represents the exact absorbing condition on the artificial boundary Γ. Here, $\tilde{\mu}_n(\cos\vartheta)=\sqrt{(2n+1)/(2n(n+1))}P_n^1(\cos\vartheta)$, $\{\tilde{\mu}_n(\cos\vartheta)\}_{n=1,2,\ldots}$ forms a complete orthonormal (with weight function $\sin\vartheta$) system of functions in the space $\mathbf{L}_2\left[(0<\vartheta<\pi)\right]$, $P_n^1(x)$ are the associated Legendre functions, $P_n(q)$ denotes a Legendre polynomial. The condition (2) is spoken of as *exact* because *any* outgoing wave described by the initial problem (1) satisfies this condition. Every outgoing wave $U(g,t)$ passes through the boundary Γ without distortions, as if it is absorbed by the domain Ω_{ext} or its boundary Γ. That is why this condition is said to be absorbing. Other approximate conditions appearing in the literature (Absorbing Boundary Conditions, Perfectly Matched Layers) are not exact in this sense, and may be insufficient for the accurate simulation of resonant structures over the relatively long time period of simulation.

2. Following the technique developed in [2], it has been proved that the original open problem (1) and the modified closed problem (with the exact absorbing condition added) are mathematically equivalent [3].

3. The standard discretization of the closed problem by the finite difference method using a uniform rectangular mesh attached to coordinates $g=\{\rho,z\}$ leads to explicit computational schemes with uniquely defined mesh functions $U(j,k,m)=U(\rho_j,z_k,t_m)$. The approximation error is $O(\bar{h}^2)$, where $\bar{h}$ is the mesh width in spatial coordinates, $\bar{l}=\bar{h}/2$ for $\theta=\max_{g\in\Omega_{\text{int}}}\left[\varepsilon(g)\right]<2$ or $\bar{l}<\bar{h}/2$ for $\theta\geq 2$ is the mesh width in time variable t; $\rho_j=j\bar{h}$, $z_k=k\bar{h}$, and $t_m=m\bar{l}$. The range of the integers $j=0,1,...,J$, $k=0,1,...K$, and $m=0,1,...M$ depends both on the size of the Ω_{int} domains and on the length of the interval $[0,T]$ of the observation time t. The condition providing uniform boundedness of the approximate solutions $U(j,k,m)$ with decreasing $\bar{h}$ and $\bar{l}$ is met (see, for example, formula (1.50) in [1]). Hence the finite-difference computational schemes are stable, and the mesh functions $U(j,k,m)$ converge to the solutions $U(\rho_j,z_k,t_m)$ of the original problem. The software based on the above described algorithm has been developed. A series of numerical results obtained with the help of this software will be presented.

4. Conclusion

A problem of efficient truncation of the computational domain in finite-difference methods is discussed for axially-symmetrical open electrodynamic structures. The original problem describing electromagnetic wave scattering on a compact axially-symmetric structure with feeding waveguides is an initial boundary-value problem formulated in an unbounded domain. The exact absorbing conditions have been derived for a spherical artificial boundary enveloping all sources and scatterers in order to truncate the computational domain and replace the original open problem by an equivalent closed one. The constructed solution has been generalized to the case of extended and remote field sources. The developed software allowed us to obtain numerical results illustrating the efficiency of our algorithm.

5. References

1. Y. K. Sirenko, S. Strom, and N. P. Yashina, "Modeling and Analysis of Transient Processes in Open Resonant Structures. New Methods and Techniques", Springer, New York, USA, 2007.

2. O. A. Ladyzhenskaya, "The boundary value problems of mathematical physics", Springer-Verlag, New York, USA, 1985.

3. O. Shafalyuk, Y. K. Sirenko and P. D. Smith, "Exact absorbing conditions for FDTD analysis of axially-symmetrical open structures", *International Conference on Electromagnetics in Advanced Applications Proceedings*, Sydney, Australia, 2010, pp. 422-425.