Plane Wave Diffraction by a Varying Impedance Discontinuity in a Perfectly Conducting Circular Waveguide

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Abstract

Diffraction of electromagnetic plane waves by a finite length cylinder with variable surface impedance which is located between two semi-infinitely long perfectly conducting cylinders is investigated rigorously. Direct Fourier transform technique is applied and a Fredholm integral equation of the second kind is determined which was solved by considering the residue contributions of the poles. Thus, the solution of the problem is reduced into the determination of infinite number of unknown coefficients satisfying an infinite system of linear algebraic equations. The influence of the varying impedance surface and the waveguide radius on the reflected and transmitted waves is presented graphically.

1 Introduction

The scattering problem of an incident plane wave which propagates along a circular cylinder waveguide constitutes one of the interesting research topics in electromagnetic theory. Because of its importance in many microwave applications such as microwave filters, the plane wave diffraction by finite, semi-finite or infinite cylinders has been extensively studied by using the Wiener-Hopf technique [1-3].

In this study, the diffraction of an electromagnetic plane wave by a finite length cylinder with variable surface impedance which is located between two semi-infinitely long perfectly conducting cylinders (see Fig-1.) is investigated rigorously. This work can also be considered as the generalization of [4] to a circular geometry. The Fourier transform of the well known Helmholtz equation with corresponding boundary conditions reduces the problem into a second kind Fredholm integral equation. The kernel of this integral equation consists of the multiplication of the Fourier transforms related to scattered field to be determined and variable impedance function, say $Z(z)$. Finally, by considering the residue contributions for the poles of the kernel function, the integral mentioned above can easily be evaluated and then, the solution of this diffraction problem is reduced into the determination of infinite number of unknown coefficients satisfying an infinite system of linear algebraic equations. After the computation of these coefficients numerically, some graphical results may be presented to show the influence of the parameters related to varying impedance surface and the waveguide radius on the reflected and transmitted waves.

2 Formulation of the Problem

Consider a cylindrical waveguide with a radius $\rho = a$, while (ρ, φ, z) being the usual cylindrical coordinates. The part $0 < z < l$ of the waveguide is characterized by a locally perturbed surface impedance denoted by $\eta(z) = Z(z)/Z_o$ with Z_o being the characteristic impedance of the free space. Let an incident acoustic plane wave with angular frequency ω, propagating in the positive z direction be given by

$$u^i(\rho, z) = e^{ikz} \tag{1}$$

where an $e^{-i\omega t}$ time factor is assumed and suppressed. k is the propagation constant which is assumed to have a small imaginary part. The lossless case can then be obtained by letting $\mathcal{I}mk \to 0$ at the end of the

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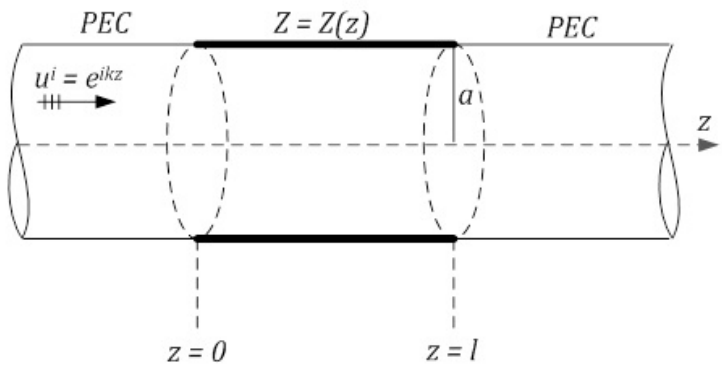

Figure 1: The geometry of the problem.

analysis. The total field $u^T(\rho, z)$ can be written as

$$u^T(\rho, z) = u(\rho, z) + u^i(\rho, z) \quad , \quad \rho \in (0, a) \ , \ -\infty < z < \infty. \tag{2}$$

$u(\rho, z)$ appearing in (2) is the diffracted field to be solved which satisfies the Helmholtz equation

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{\partial^2}{\partial z^2} + k^2\right] u(\rho, z) = 0 \quad , \quad -\infty < z < \infty. \tag{3}$$

The Fourier transform of the above equation yields

$$\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right) + K^2(\alpha)\right] \hat{u}(\rho, \alpha) = 0 \tag{4}$$

with

$$\hat{u}(\rho, \alpha) = \int_{-\infty}^{\infty} u(\rho, z)\, e^{i\alpha z} dz. \tag{5}$$

In (4) $K(\alpha)$ denotes the square-root function

$$K(\alpha) = \sqrt{k^2 - \alpha^2} \tag{6}$$

which is defined in the complex $\alpha-$plane, cut along $\alpha = k$ to $\alpha = k + i\infty$ and $\alpha = -k$ to $\alpha = -k - i\infty$, such that $K(0) = k$. The general solution of the equation (4) yields

$$\hat{u}(\rho, \alpha) = -A(\alpha)\frac{J_o(K\rho)}{KJ_1(Ka)} \tag{7}$$

where $A(\alpha)$ is the unknown spectral coefficient to be solved under the boundary and edge conditions given below:

$$\frac{\partial u^T(a, z)}{\partial\rho} = 0 \quad , \quad z \in (-\infty, 0) \cup (l, \infty) \tag{8}$$

$$\frac{\partial u^T(a, z)}{\partial\rho} - ik\eta(z)\, u^T(a, z) = 0 \quad , \quad z \in (0, l) \tag{9}$$

$$u(\rho, z) = \mathcal{O}\left(\sqrt{z}\right) \quad , \quad z \to 0, l \tag{10}$$

In (9) $\eta(z)$ is assumed to be

$$\eta(z) = \begin{cases} 0 & , \quad z < 0 \\ \eta_2(z) & , \quad 0 < z < l \\ 0 & , \quad z > l \end{cases} \tag{11}$$

while $\eta_2(z)$ stands for

$$\eta_2(z) = \eta_j \quad , \quad z \in (b_{j-1}, b_j) \quad , \quad j = 1, 2, ..., n \tag{12}$$

with $b_o = 0$ and $b_n = l$. The Fourier transform of (8) yields

$$A(\alpha) = P_1(\alpha) \tag{13}$$

with

$$P_1(\alpha) = \int_0^l \frac{\partial u(a,z)}{\partial \rho} e^{i\alpha z} d\alpha. \tag{14}$$

On the other hand, by applying the Fourier transform to (9), one gets

$$P_1(\alpha) - ik \int_0^l \eta(z) u(a,z) e^{i\alpha z} d\alpha = ik \int_0^l \eta(z) u^i(a,z) e^{i\alpha z} d\alpha. \tag{15}$$

Taking into account (11), one can express the integral at the left-hand side of the above equation as

$$\int_0^l \eta(z) u(a,z) e^{i\alpha z} d\alpha = \int_{-\infty}^{\infty} \eta(z) u(a,z) e^{i\alpha z} d\alpha. \tag{16}$$

Defining the convolution integral in $\alpha-$domain

$$f(\alpha) * g(\alpha) = \int_{\mathcal{L}} f(\tau - \alpha) g(\tau) d\tau \tag{17}$$

(15) becomes

$$P_1(\alpha) - ik \{\hat{\eta}(\alpha) * \hat{u}(\rho, \alpha)\} = k \sum_{j=1}^{n} \eta_j \frac{e^{ib_j(\alpha+k)} - e^{ib_{j-1}(\alpha+k)}}{(\alpha+k)} \tag{18}$$

where we put

$$\int_0^l \eta(z) u^i(a,z) e^{i\alpha z} d\alpha = \sum_{j=1}^{n} \eta_j \frac{e^{ib_j(\alpha+k)} - e^{ib_{j-1}(\alpha+k)}}{i(\alpha+k)}. \tag{19}$$

The Fourier transform of $\eta(z)$ is

$$\hat{\eta}(\alpha) = \int_{-\infty}^{\infty} \eta(z) e^{i\alpha z} d\alpha = \int_0^l \eta(z) e^{i\alpha z} d\alpha = \sum_{j=1}^{n} \eta_j \frac{e^{i\alpha b_j} - e^{i\alpha b_{j-1}}}{i\alpha}. \tag{20}$$

Substituting this in the convolution of $\hat{\eta}(\alpha) * \hat{u}(\rho, \alpha)$, one obtains the integral

$$\hat{\eta}(\alpha) * \hat{u}(\rho, \alpha) = -\sum_{j=1}^{n} \eta_j \int_{\mathcal{L}} \frac{\left[e^{i(\tau-\alpha)b_j} - e^{i(\tau-\alpha)b_{j-1}}\right]}{i(\tau-\alpha)} P_1(\tau) \frac{J_o[K(\tau) a]}{K(\tau) J_1[K(\tau) a]} d\tau. \tag{21}$$

The evaluation of this integral gives

$$\hat{\eta}(\alpha) * \hat{u}(\rho, \alpha) = \frac{2\pi}{a} \sum_{j=1}^{n} \eta_j \sum_{m=0}^{\infty} \frac{\left[e^{i(\alpha_m-\alpha)b_{j-1}} - e^{i(\alpha_m-\alpha)b_j}\right]}{\alpha_m(\alpha-\alpha_m)} P_1(\alpha_m) \tag{22}$$

where α_m's are the simple poles of the integrand which satisfy

$$K_m J_1(K_m a) = 0 \quad , \quad m = 0, 1, 2, ... \tag{23}$$

$$\alpha_m = \sqrt{k^2 - K_m^2}. \tag{24}$$

Taking into account (22), $P_1(\alpha)$ is solved as

$$P_1(\alpha) = \frac{2\pi ik}{a} \sum_{j=1}^{n} \eta_j \sum_{m=0}^{\infty} \frac{\left[e^{i(\alpha_m-\alpha)b_{j-1}} - e^{i(\alpha_m-\alpha)b_j}\right]}{\alpha_m(\alpha-\alpha_m)} P_1(\alpha_m) + k \sum_{j=1}^{n} \eta_j \frac{e^{ib_j(\alpha+k)} - e^{ib_{j-1}(\alpha+k)}}{(\alpha+k)}. \tag{25}$$

The coefficients $P_1(\alpha_m)$'s are determined by solving the linear system of algebraic equations

$$P_1(\alpha_n) - \frac{2\pi ik}{a} \sum_{j=1}^{n} \eta_j \sum_{m=0}^{\infty} \frac{\left[e^{i(\alpha_m-\alpha_n)b_{j-1}} - e^{i(\alpha_m-\alpha_n)b_j}\right]}{\alpha_m(\alpha_n-\alpha_m)} P_1(\alpha_m) = k \sum_{j=1}^{n} \eta_j \frac{e^{ib_j(\alpha_n+k)} - e^{ib_{j-1}(\alpha_n+k)}}{(\alpha_n+k)}. \tag{26}$$

3 References

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