Electromagnetic Oscillations in a Cylindrical Cavity Resonator Filled with a Nonlinear Nondispersive Medium

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Abstract

A new method for deriving exact axisymmetric solutions of the Maxwell equations in a nonlinear nondispersive medium lacking a center of inversion is proposed. Using this method, an implicit solution that describes free electromagnetic oscillations in a cylindrical cavity resonator filled with such a medium is found. Implications of the results obtained and their further generalization to more complex cases are discussed.

1. Introduction

Despite the development of computational techniques, finding new, physically important exact solutions of nonlinear partial differential equations describing electromagnetic fields in nonlinear media seems very topical. In most papers on the subject, nonlinear waves in unbounded regions or oscillations in the simplest lumped nonlinear resonators are considered. At the same time, the behavior of electromagnetic fields in distributed nonlinear resonance systems such as cavity resonators filled with nonlinear media remains poorly studied. In this work, we present a new method for constructing exact axisymmetric solutions of the Maxwell equations in a nonlinear nondispersive medium. It is assumed that the medium lacks a center of inversion and the dependence of the electric displacement on the electric field can be approximated by an exponential function. Using the proposed method, we analyze free electromagnetic oscillations in a cylindrical cavity resonator filled with such a medium.

2. Formulation of the Problem and Basic Equations

Consider electromagnetic fields in a loss-free nonmagnetic medium. We assume that the medium possesses an axis of symmetry, hereafter taken as the z axis of a cylindrical coordinate system (r, ϕ, z). If the fields are independent of ϕ and z, the Maxwell equations admit solutions in which only the E_z and H_ϕ components are nonzero (E waves with respect to the symmetry axis). Restricting ourselves to consideration only of such solutions, we will also neglect dispersive effects and suppose that the relation of the displacement D_z to the electric field E_z is local in space and time. Denoting $E_z(r,t)$, $D_z(r,t)$, and $H_\phi(r,t)$ as E, D, and H, respectively, we can write equations for these functions as

$$\frac{\partial H}{\partial r} + \frac{H}{r} = \varepsilon(E)\frac{\partial E}{\partial t}, \quad \frac{\partial E}{\partial r} = \mu_0 \frac{\partial H}{\partial t}, \tag{1}$$

where $\varepsilon(E) = dD/dE$ and μ_0 is the permeability of free space. System (1) can be reduced to the nonlinear wave equation

$$\frac{\partial^2 E}{\partial r^2} + \frac{1}{r}\frac{\partial E}{\partial r} = \mu_0 \frac{\partial}{\partial t}\left(\varepsilon(E)\frac{\partial E}{\partial t}\right). \tag{2}$$

System (1) and equation (2) are integrated exactly if the function $\varepsilon(E)$ is chosen in the form

$$\varepsilon(E) = \epsilon_0 \varepsilon_1 \exp(\alpha E), \tag{3}$$

where ϵ_0 is the permittivity of free space, and ε_1 and α are certain constants. The quantity D can be represented as

$$D = D_0 + \epsilon_0 \varepsilon_1 \alpha^{-1}\left[\exp(\alpha E) - 1\right] = D_0 + \epsilon_0 \varepsilon_1 \left(E + \alpha E^2/2 + \ldots\right), \tag{4}$$

where $D_0 = D(E=0) = \text{const}$. Equation (4) shows that even powers of E are present in the series expansion of D. Thus, the medium for which the dependence $\varepsilon(E)$ is approximated by function (3) does not possess a center

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of inversion [1]. This is inherent in, e.g., uniaxial pyroelectric and ferroelectric crystals, provided that the z axis is aligned with the crystallographic symmetry axis. The case where $D_0 \neq 0$ corresponds to the presence of spontaneous polarization. Along with the medium properties, another factor leading to lack of a center of inversion can be the presence of a strong external electric field [1]. For example, let an isotropic medium in which $D = \epsilon_0 \varepsilon_1 E_z + \chi^{(3)} E_z^3$, where $\chi^{(3)} = \text{const}$, be placed in a uniform static electric field $\boldsymbol{E} = E_0 \hat{\boldsymbol{z}}_0$. Representing the total field as $E_z = E_0 + E$, we obtain the term proportional to E^2 in the expansion of D. Thus, with appropriately chosen constants D_0, ε_1 and α, formula (4) correctly describes dielectric properties of media lacking a center of inversion in the case of weak nonlinearity where we can retain only the quadratic (in E) correction term to the linear dependence of D on E.

3. Method of Solution

Let us use the following ansatz in system (1):

$$E = \alpha^{-1}(u - 2\xi), \quad H = \varepsilon_1^{1/2} (Z_0 \alpha)^{-1} e^{-\xi} (v - 2\eta), \tag{5}$$

where $\xi = \ln(r/r_0)$, $\eta = t(\epsilon_0 \varepsilon_1 \mu_0)^{-1/2}/r_0$, $Z_0 = (\mu_0/\epsilon_0)^{1/2}$, and r_0 is an arbitrary constant with the dimension of length. In the new variables, we have

$$\frac{\partial u}{\partial \xi} = \frac{\partial v}{\partial \eta}, \quad \frac{\partial v}{\partial \xi} = e^u \frac{\partial u}{\partial \eta}. \tag{6}$$

System (6) has particular solutions in which one of the functions u and v can be expressed in terms of the other:

$$u = F(\xi \pm \eta \, e^{-u/2}), \quad v = \pm 2(e^{u/2} - 1). \tag{7}$$

Here, F is an arbitrary differentiable function. Similar solutions, which are analogous to the Riemann solutions in fluid mechanics, have been obtained in [2]. However, it can readily be verified that ansatz (5) does not make it possible to arrive at physically admissible solutions for E_z and H_ϕ on the basis of (7) in our case [3, 4], and somewhat another approach should be used. The approach is based on the application of a hodograph transformation for seeking solutions for which the Jacobian $D(u,v)/D(\xi,\eta)$ is nonzero. Using u and v as independent variables, from (6) we obtain

$$\frac{\partial \eta}{\partial v} = \frac{\partial \xi}{\partial u}, \quad \frac{\partial \eta}{\partial u} = e^u \frac{\partial \xi}{\partial v}. \tag{8}$$

Excluding η from (8) and making the replacement $w = 2\exp(u/2)$, we arrive at the equation

$$\frac{\partial^2 \xi}{\partial w^2} + \frac{1}{w} \frac{\partial \xi}{\partial w} - \frac{\partial^2 \xi}{\partial v^2} = 0. \tag{9}$$

A remarkable symmetry property of system (1) with exponential nonlinearity is that it is reduced to linear wave equation (9) for cylindrical waves by application of the above-described substitutions and the hodograph transformation. However, initial and boundary conditions for the fields E and H in the new variables ξ and η can become much more complicated. This may cause the necessity of numerically solving even linear equation (9). Nevertheless, it is possible to propose a comparatively simple analytical method which permits one to find physically admissible exact solutions of system (1). The idea of the method consists in the following. At first, one should find an analytical solution to the problem of propagation of cylindrical E waves in a medium with the linear dependence $D_z = D_0 + \epsilon_0 \varepsilon_1 E_z$. Assume that such a solution is known and we have the functions E and H satisfying the linear field equations and the specified initial and boundary conditions. The characteristic spatial scale determined by these conditions for the problem considered will be denoted by a. We also introduce the dimensionless variables $\rho = r/a$ and $\tau = t(\epsilon_0 \varepsilon_1 \mu_0)^{-1/2}/a$. Then it is convenient to represent the solution of the linear problem in the form

$$E \equiv E_z = \mathcal{E}(\rho, \tau), \quad H \equiv H_\phi = Z_0^{-1} \varepsilon_1^{1/2} \, \mathcal{H}(\rho, \tau), \tag{10}$$

where the functions $\mathcal{E}$ and $\mathcal{H}$ satisfy the system

$$\frac{\partial \mathcal{H}}{\partial \rho} + \frac{\mathcal{H}}{\rho} = \frac{\partial \mathcal{E}}{\partial \tau}, \quad \frac{\partial \mathcal{E}}{\partial \rho} = \frac{\partial \mathcal{H}}{\partial \tau}. \tag{11}$$

We write the quantities ξ and η as

$$\xi = C\mathcal{E}(w, v) + \ln \frac{w}{2}, \quad \eta = \frac{C}{2} \, w \, \mathcal{H}(w, v) + \frac{v}{2}, \tag{12}$$

where C is an arbitrary constant. It can be verified by straightforward differentiation that functions (12) satisfy system (8). Using formulas (5), we can pass to the initially used quantities r, t, E, and H in (12). Putting $C = -\alpha/2$ and $r_0 = 2a$ ensures that the resulting solution will go into solution (10) in the linear case. Bearing this in mind, after some simple algebra we obtain

$$E = \mathcal{E}\left(\rho\, e^{\alpha E/2}, \tau + \alpha Z_0 \rho H/(2\sqrt{\varepsilon_1})\right), \quad H = Z_0^{-1}\varepsilon_1^{1/2}\, e^{\alpha E/2}\mathcal{H}\left(\rho\, e^{\alpha E/2}, \tau + \alpha Z_0 \rho H/(2\sqrt{\varepsilon_1})\right). \tag{13}$$

These expressions give an exact solution of system (1) in implicit form and describe axisymmetric electromagnetic fields in the nonlinear medium considered. For the known functions $\mathcal{E}$ and $\mathcal{H}$, which are determined by solving the linear problem, and given values of ρ and τ, formulas (13) represent a system of two transcendental equations in E and H. In the limit $\alpha \to 0$, the solution obtained goes into solution (10) of the linear problem, but, generally, corresponds to somewhat different initial or boundary conditions compared with those satisfied by functions (10).

To make an application of the above formulation, in what follows we solve the problem of free oscillations of an axisymmetric electromagnetic field in a cylindrical resonator with a nonlinear medium lacking a center of inversion.

4. Free Electromagnetic Oscillations in a Cylindrical Resonator

Consider a cavity resonator, which is a perfectly conducting circular cylinder of radius a and height L. We assume that the z axis is aligned with the cavity axis and the perfectly conducting end walls of the cavity are at $z = 0$ and $z = L$. In the case where the cavity resonator is filled with a linear medium having the permittivity $\varepsilon = \epsilon_0\varepsilon_1 = \text{const}$, E_{0n0} (TM_{0n0}) modes can exist in the cavity. The E_z and H_ϕ components, which are nonzero in these modes, are independent of ϕ and z, and are described by the solutions of system (11) with the boundary conditions

$$E_z|_{r=a} \equiv \mathcal{E}(1,\tau) = 0, \quad |\, E_z|_{r=0}\,| \equiv |\mathcal{E}(0,\tau)| < \infty. \tag{14}$$

Such solutions are well-known and their derivation can be found elsewhere. Substituting the functions $\mathcal{E}$ and $\mathcal{H}$, which describe the E_{0n0} modes in the linear case, into Eqs. (13), we obtain the solution of nonlinear equations (1) in the form

$$E = A\, J_0(\kappa_n\, \rho\, e^{\alpha E/2})\, \cos(\kappa_n\theta), \quad H = -AZ_0^{-1}\varepsilon_1^{1/2}\, e^{\alpha E/2} J_1(\kappa_n\, \rho\, e^{\alpha E/2})\, \sin(\kappa_n\theta), \tag{15}$$

where $\theta = \tau + \alpha Z_0 \rho H/(2\sqrt{\varepsilon_1})$, J_m is a Bessel function of the first kind of order m, κ_n is the nth root of the equation $J_0(\kappa) = 0$, and A is an arbitrary amplitude factor. Note that the field $E = 0$ satisfies the transcendental equations (15) for any τ if $\rho = r/a = 1$. Therefore, the boundary conditions (14) remain valid for the implicit function $E(\rho,\tau)$ defined by (15). Thus, formulas (15) yield an exact solution of the nonlinear boundary value problem for system (1) under conditions (14) and describe free electromagnetic oscillations in a cylindrical cavity filled with a nonlinear medium.

Implicit solutions E and H given by (15) and corresponding to a certain index n are periodic functions of time t with period $T_n = 2\pi/\omega_n$, where $\omega_n = \kappa_n(\epsilon_0\varepsilon_1\mu_0)^{-1/2}a^{-1}$ is an eigenfrequency of the E_{0n0} mode. Along with the fundamental frequency ω_n for each n, the Fourier time series expansions of the functions E and H also contain terms at the multiple frequencies $l\omega_n$, where l is an integer. The contribution of harmonics with $l \geq 2$ determines the role of nonlinear effects which manifest themselves as deviations of the quantities E and H from their values corresponding to the E_{0n0} mode in a cavity with $\varepsilon = \epsilon_0\varepsilon_1 = \text{const}$ in the linear case ($\alpha = 0$).

Now present some results of calculations of the quantities E and H by formulas (15). Figure 1(a) shows snapshots of the normalized field components E/A and Z_0H/A in the lowest mode ($n = 1$ and $\kappa_1 \simeq 2.4$) as functions of ρ at fixed instants of time τ. Figures 1(b) and 1(c) show the oscillograms of the field components at the points $\rho = 0.2$ and $\rho = 0.7$ for $n = 1$. Similar curves for a mode with $n = 2$ and $\kappa_2 \simeq 5.5$ are presented in Figs. 1(d)–1(f). Note that all panels in Fig. 1 were plotted for $\alpha A = 0.5$ and $\varepsilon_1 = 2$. The presented plots show that the nonlinear effects become more pronounced with increasing n and depend significantly on ρ, i.e., location of the observation point inside the cavity. For example, the field oscillograms in Fig. 1(b) are analogous to those in the linear case. However, in Fig. 1(f) we see that the field E varies at the frequency ω_2, while the field H, at the second harmonic $2\omega_2$. For the higher modes with $n > n^*$, where n^* is an integer depending on the parameter αA, the functions $E(\rho,\tau)$ and $H(\rho,\tau)$ determined by (15) become ambiguous in a certain domain of values of ρ and τ. Since such behavior is

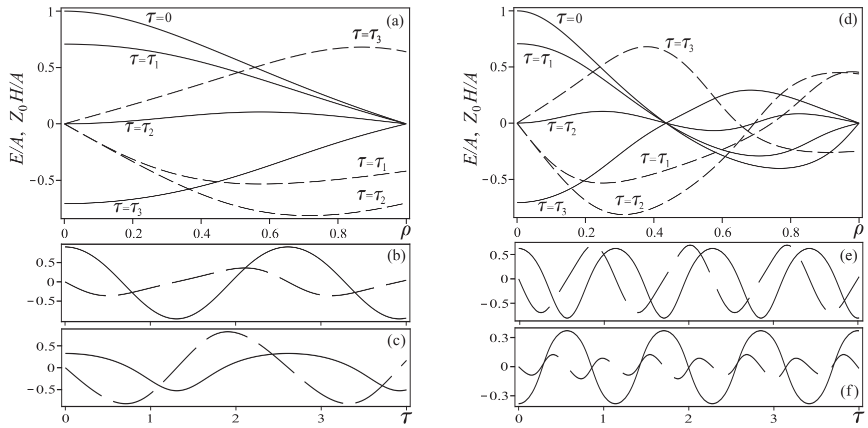

Figure 1. (a) Electric and magnetic fields as functions of ρ (solid and dashed lines, respectively) in the $n = 1$ mode at times $\tau_1 = \pi/(4\kappa_1)$, $\tau_2 = \pi/(2\kappa_1)$, and $\tau_3 = 5\pi/(4\kappa_1)$ and the oscillograms of the fields at (b) $\rho = 0.2$ and (c) $\rho = 0.7$. (d) The same as in Fig. 1(a), but for the $n = 2$ mode at $\tau_1 = \pi/(4\kappa_2)$, $\tau_2 = \pi/(2\kappa_2)$, and $\tau_3 = 5\pi/(4\kappa_2)$, and the corresponding field oscillograms at (e) $\rho = 0.2$ and (f) $\rho = 0.7$.

not physically admissible, one should expect field discontinuities at the ambiguity points. The time dependences of E and H can then be discontinuous (relaxation) oscillations, and the solutions (15) obtained without allowance for dispersion become inapplicable. However, for weak nonlinearity ($|\alpha A| \ll 1$), the number n^* is large (e.g., $n^* = 9$ for $\alpha A = 0.5$) and solutions (15) for $n < n^*$ are single-valued continuous functions of coordinates and time.

Finally, we note that although only the case of a cylindrical cavity was discussed in the above analysis, the approach presented can be extended to other resonators, e.g., coaxial resonators filled with a nonlinear medium. The corresponding results are not shown here for brevity.

5. Conclusion

In conclusion, we can state that the proposed method makes it to possible to easily generate various physically interesting solutions of nonlinear system (1), starting from the corresponding solutions of linear field equations. Hence, this method may have significant advantages over the direct numerical solution of that system. The exact solutions found for free electromagnetic oscillations in a cavity with a nonlinear medium seem to be of great practical interest and can be very useful for analysis of, e.g., ferroelectric resonators.

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7. References

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