Homogenization of a Nonlocal Electrostatic Equation

Niklas Wellander
Swedish Defence Research Agency, FOI
P.O. Box 1165, SE-58111 Linköping (Sweden)
and
Lund University
P.O. Box 118, SE-221 00 Lund (Sweden)
niklas.wellander@foi.se

Abstract

We find the effective (homogenized) properties of a composite (a heterogeneous material) supplied with spatially non-local constitutive relations. We homogenize an electrostatic equation in a periodic setting. The current density is given as a spatial convolution of the electric field with a conductivity kernel. It turns out that the homogenized equation also has a nonlocal constitutive relation if we do not scale the non-localness. However, if we decrease the neighborhood which influence the current density simultaneously as we make the fine structure scale finer and finer then we obtain a constitutive relation which is local.

1 Introduction

Multiple scales phenomena are ubiquitous, ranging from mechanical properties of wood, turbulent flow in gases and fluids, combustion, remote sensing of earth to wave propagation or heat conduction in composite materials. The obstacle with multi-scale problems is that they, due to limited primary memory even in the largest computational clusters, can not easily be modeled in standard numerical algorithms. Usually we are not even interested in the fine scale information in the processes. However, the fine scale properties are important for the macroscopic, effective, properties of for example a fiber composite. Attempts to find effective properties of composites dates back more than hundred years, e.g. see [1-3]. One way to find effective properties is to introduce a fine scale parameter in the corresponding governing equations (modeling fast oscillating coefficients) and then study the asymptotic behavior of the sequence of solutions, and equations, when the fine scale parameter tends to zero. The limit yields the homogenized equations, that have constant coefficients (corresponding to homogeneous material properties). The discipline of partial differential equations dealing with such issues is called homogenization theory. As an introduction to the theory of composites we recommend the monograph [4]. For mathematical treatments we recommend [5] and [6].

As a model we will consider two non-local elliptic problems. The physical problem in mind is a nonlocal electrostatic equation for a periodic composite. This is an elliptic problem with spatial convolution of the electric field with a conductivity kernel, which consists of a periodic part multiplied with a localizing function. The localizer gives a finite contribution to the current density when convoluted with the electric fields in the neighborhood of the observation point.

2 Assumptions and weak formulation

The domain, Ω, is assumed to be a bounded subset of $\mathbb{R}^n$, $n \in \mathbb{N}$ with a Lipshitz boundary $\partial\Omega$. We assume the current density is given by a spatial convolution of the electric field with a nonlocal kernel $\mathbf{K}$ which gives the current density contribution at a point due to the electric field in the neighborhood of $\boldsymbol{x}$,

$$\boldsymbol{J}(\boldsymbol{x}, \nabla\phi) = \boldsymbol{J}(\boldsymbol{x}) = \int_\Omega \mathbf{K}(\boldsymbol{x} - \boldsymbol{\xi})\nabla\phi(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}. \tag{1}$$

The kernel maps electric fields to current densities ($\mathbb{R}^n \to \mathbb{R}^n$) and decays monotonically for large arguments. To model the fine scale structure in a heterogeneous material we introduce the fine scale parameter $\varepsilon > 0$. The scaled

978-1-4244-6051-9/11/$26.00 ©2011 IEEE

current density is given by

$$\boldsymbol{J}^\varepsilon(\boldsymbol{x}) = \int_\Omega \mathbf{K}^\varepsilon(\boldsymbol{x}-\boldsymbol{\xi})\nabla\phi^\varepsilon(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \tag{2}$$

where ϕ^ε is the electric potential. We integrate over the support of $\mathbf{K}^\varepsilon$ which overlaps Ω (which has to be taken into account close to the boundary $\partial\Omega$). The static equation reads

$$\begin{cases} -\nabla\cdot\boldsymbol{J}^\varepsilon(\boldsymbol{x}) = f^\varepsilon(\boldsymbol{x}) & \boldsymbol{x}\in\Omega \\ \phi^\varepsilon|_{\partial\Omega} = 0 \end{cases} \tag{3}$$

where f^ε is some given current density source bounded in $L^2(\Omega)$ which converges strongly to f in $H^{-1}(\Omega)$ when $\varepsilon\to 0$. Equation (3) is to be understood in the weak sense, *i.e.*,

$$\int_\Omega \boldsymbol{J}^\varepsilon(\boldsymbol{x})\cdot\nabla\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_\Omega f^\varepsilon(\boldsymbol{x})\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \qquad \forall\psi\in H_0^1(\Omega) \tag{4}$$

We introduce the scaled bilinear form

$$a^\varepsilon(\phi,\psi) = \int_\Omega\int_\Omega \mathbf{K}^\varepsilon(\boldsymbol{x}-\boldsymbol{\xi})\nabla\phi(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}\cdot\nabla\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \tag{5}$$

Equation (4) can now be restated in the following weak formulation. Find $\phi^\varepsilon\in H_0^1(\Omega)$ such that

$$a^\varepsilon(\phi^\varepsilon,\psi) = \int_\Omega f^\varepsilon(\boldsymbol{x})\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \qquad \forall\psi\in H_0^1(\Omega) \tag{6}$$

We will assume that the kernel $\mathbf{K}$ is such that there exist constants $C_1, C_2 > 0$ such that

$$|a^\varepsilon(\phi,\psi)| \le C_1\|\nabla\phi\|_{L^2(\Omega;\mathbb{R}^n)}\|\nabla\psi\|_{L^2(\Omega;\mathbb{R}^n)} \tag{7}$$

$$C_2\|\nabla\phi\|^2_{L^2(\Omega;\mathbb{R}^n)} \le a^\varepsilon(\phi,\phi) \tag{8}$$

for all $\phi,\psi\in H_0^1(\Omega)$. Equation (6) has a unique solution $\phi^\varepsilon\in H_0^1(\Omega)$ for each $\varepsilon>0$ [7].

The main question to be answered is: Which equation with constand coefficients has a solution that is the best possible approximation of the solution of equation (6) when ε is small? To be able to answer this question we need to find the limit of the bilinear form when $\varepsilon\to 0$. The first step is to establish a priori estimates of the sequence of solutions. We have the standard a priori estimate of the solutions of (6)

$$\|\phi^\varepsilon\|_{H_0^1(\Omega)} \le C \tag{9}$$

uniformly with respect to $\varepsilon>0$.

3 Homogenization

In the first case the non-localness is conserved in the homogenized equation. In the second we also scale the size of the domain of influence of the kernel in the non-local constitutive relation, which gives a local constitutive relation in the homogenized equation.

3.1 Case I, Non-vanishing non-localness

Assume that $\mathbf{K}$ satisfies some certain good regularity properties (not specified here). In this paper we assume it is given by the product of a periodic function with the standard mollifier

$$\mathbf{K}(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} C\boldsymbol{\sigma}(\boldsymbol{y})\exp\left(\frac{1}{\left|\frac{\boldsymbol{x}}{r}\right|^2-1}\right) & , \qquad |\boldsymbol{x}|<r. \\ 0 & , \qquad |\boldsymbol{x}|\ge r \end{cases} \tag{10}$$

where r is the radius of the non-local influence zone, $\boldsymbol{\sigma}$ is the conductivity associated with the non-locality, it is assumed to be Y-periodic, *i.e.*, $\boldsymbol{\sigma}(\boldsymbol{y}+\boldsymbol{e}) = \boldsymbol{\sigma}(\boldsymbol{y})$ for all $\boldsymbol{y} \in]0,1[^n$, and $C > 0$ is a constant. The scaled kernel reads

$$\mathbf{K}^\varepsilon(\boldsymbol{x}) = \mathbf{K}\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\right) = \begin{cases} C\boldsymbol{\sigma}\left(\frac{\boldsymbol{x}}{\varepsilon}\right)\exp\left(\frac{1}{\left|\frac{\boldsymbol{x}}{r}\right|^2 - 1}\right) & , \quad |\boldsymbol{x}| < r. \\ 0 & , \quad |\boldsymbol{x}| \geq r \end{cases} \tag{11}$$

Moreover, the conductivity $\boldsymbol{\sigma}$ satisfies the coercivity condition

$$\boldsymbol{\sigma}\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq c_1 |\boldsymbol{\xi}|^2 \tag{12}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^n$, almost every $\boldsymbol{x} \in \Omega$, and it is bounded, *i.e.*, $\boldsymbol{\sigma} \in L^\infty(\Omega; \mathbb{R}^{n\times n})$.

Theorem 1 (Homogenization, non-vanishing non-localness) *Let* $\{\phi^\varepsilon\}$ *be a sequence of solutions to* (6) *where the kernel in the bilinear form* (5) *is given by* (11)*. The sequence* $\{\phi^\varepsilon\}$ *converges weakly in* $H_0^1(\Omega)$ *to* $\phi \in H_0^1(\Omega)$*, the unique solution of the Homogenized Problem*

$$-\nabla \cdot \int_{\Omega \cap \operatorname{supp} \boldsymbol{\sigma}_h(\boldsymbol{x}-\cdot)} \boldsymbol{\sigma}_h(\boldsymbol{x}-\boldsymbol{z})\, \nabla\phi(\boldsymbol{z})\, \mathrm{d}\boldsymbol{z} = f(\boldsymbol{x}), \tag{13}$$

almost everywhere in Ω*, where the homogenized conductivity is given by*

$$\boldsymbol{\sigma}_h(\boldsymbol{x}) = \int_{T^n} \mathbf{K}(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}\boldsymbol{y} = \int_{T^n} C\boldsymbol{\sigma}(\boldsymbol{y}) \exp\left(\frac{1}{\left|\frac{\boldsymbol{x}}{r}\right|^2 - 1}\right) \mathrm{d}\boldsymbol{y} \tag{14}$$

3.2 Case II, Vanishing non-localness

In this case we will use the same kernel, but we will scale both variables, *i.e.*, let

$$\mathbf{K}(\boldsymbol{y}) = \begin{cases} C\boldsymbol{\sigma}(\boldsymbol{y})\exp\left(\frac{1}{\left|\frac{\boldsymbol{y}}{r}\right|^2 - 1}\right) & , \quad |\boldsymbol{y}| < r. \\ 0 & , \quad |\boldsymbol{y}| \geq r \end{cases} \tag{15}$$

where $r > 1$ is the radius of the non-local influence zone, and $C > 0$ is a constant and scale the kernel as

$$\mathbf{K}^\varepsilon(\boldsymbol{x}) = \varepsilon^{-n}\mathbf{K}\left(\frac{\boldsymbol{x}}{\varepsilon}\right) = \begin{cases} \varepsilon^{-n} C\boldsymbol{\sigma}\left(\frac{\boldsymbol{x}}{\varepsilon}\right)\exp\left(\frac{1}{\left|\frac{\boldsymbol{x}}{\varepsilon r}\right|^2 - 1}\right) & , \quad |\boldsymbol{x}| < \varepsilon r. \\ 0 & , \quad |\boldsymbol{x}| \geq \varepsilon r \end{cases} \tag{16}$$

The assumptions for Case I applies.

Theorem 2 (Homogenization, vanishing non-localness) *Let* $\{\phi^\varepsilon\}$ *be a sequence of solutions to* (6) *where the kernel in the bilinear form* (5) *is given by* (16)*. The sequence* $\{\phi^\varepsilon\}$ *converges weakly in* $H_0^1(\Omega)$ *to* $\phi \in H_0^1(\Omega)$*, the unique solution of the Homogenized Problem*

$$-\nabla \cdot \boldsymbol{\sigma}_h \nabla\phi(\boldsymbol{x}) = f(\boldsymbol{x}), \tag{17}$$

almost everywhere in Ω*, where the homogenized conductivity is given by*

$$\boldsymbol{\sigma}_h = \int_{|\boldsymbol{y}|<r} \mathbf{K}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} = \int_{|\boldsymbol{y}|<r} C\boldsymbol{\sigma}(\boldsymbol{y}) \exp\left(\frac{1}{\left|\frac{\boldsymbol{y}}{r}\right|^2 - 1}\right) \mathrm{d}\boldsymbol{y} \tag{18}$$

The homogenized problems (13) and (17) has each a unique solution in $H_0^1(\Omega)$ [7].

The proofs are given in [8] and utilize a two-scale Fourier transform [8-11] which captures coarse and fine scales in functions containing well separated scales.

4 Remarks and conclusions

The localization of the constitutive relation for Case II in (17) can equally be obtained by multiplying the kernel in (10) with r^{-1} and sending $r \to 0$ either before sending $\varepsilon \to 0$ or after. Introducing a spatially local contribution in the constitutive relations will somewhat complicate the analysis, but it is doable. An effect that we have not taken into account is the influence of the boundary $\partial\Omega$. In real life, e.g. for wave propagation in cases the wavelength is on the same order as the material periodicity, we expect the nonlocal constitutive relation to depend on the distance to the boundary. We conclude that spatially nonlocal constitutive relations are particularly easy to homogenize since we need only to integrate the kernel over the fast variable.

5 References

1. M. Faraday, 1965 *Experimental researches in electricity : in three volumes bound as two*, Dover Publications, New York, 1965.

2. J. C. Maxwell, *A Treatise on Electricity and Magnetism*, Dover Publications, New York, 1954.

3. L. Rayleigh, ”On the influence of obstacles arranged in rectangular order upon the properties of the medium,” *Philosophical Magazine* **34** 1892, pp. 481–502.

4. G. W. Milton, *The Theory of Composites*, Cambridge University Press, Cambridge, U.K., 2002.

5. A. Bensoussan, J. L. Lions, and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures*, Vol. 5 of *Studies in Mathematics and its Applications*, North-Holland, Amsterdam, 1978.

6. D. Cioranescu and P. Donato *An Introduction to Homogenization*, Oxford, University Press, 1999.

7. L. C. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, Rhode Island, 1998.

8. N. Wellander, ”Homogenization of Nonlocal Electrostatic Equations by Means of the Two-Scale Fourier Transform,” *Fourier Transforms, Theory and Applications/Book 1*, Ed: Goran Nikolic, INTECH, 2011, ISBN 978-953-307-473-3, 2011, in press.

9. N. Wellander, ”Review of contemporary homogenization methods,” *2004 URSI EMTS International Symposium on Electromagnetic Theory*, Pisa, Italy, 2004.

10. N. Wellander, ”Periodic homogenization in Fourier space,” *ICIAM07, 6th International Congress on Industrial and Applied Mathematics*, 2007.

11. N. Wellander, ”The two-scale Fourier transform approach to homogenization; periodic homogenization in Fourier space,” *Asymptot. Anal.* 62 (1-2), 2009, pp. 1–40.