The Class of Decomposable Media in Four-Dimensional Representation

I.V. Lindell[1], *L. Bergamin*[2], and *A. Favaro*[3]
[1] Aalto University, School of Electrical Engineering, Department of Radio Science and Engineering, P.O. Box 13000, FI-00076 AALTO, Espoo, Finland, ismo.lindell@aalto.fi
[2] KB&P GmbH, Fliederweg 10, Bern, Switzerland, luzi.bergamin@kbp.ch
[3] Imperial College London, Department of Physics, Prince Consort Road, SW7 2AZ, London, UK, alberto.favaro04@imperial.ac.uk

Abstract

The well-known TE/TM decomposition of time-harmonic electromagnetic fields in uniaxial anisotropic media is generalized in terms of four-dimensional differential-form formalism by requiring that the field two-form satisfies an orthogonality condition with respect to two given bivectors. Conditions for the electromagnetic medium in which such a decomposition is possible are derived and found to define three subclasses of media. Two of the subclasses are defined by medium dyadics which satisfy certain equation of the second order while while the medium dyadic defining the third subclass satisfies an equation of the first order. One of the subclasses is previously known through an analysis applying three-dimensional Gibbsian dyadics. Dispersion equations for plane waves are derived and the corresponding eigenpolarizations are found for all three subclasses.

1 Introduction

It is well known that in a uniaxially anisotropic medium defined by permittivity and permeability dyadics of the form

$$\overline{\overline{\epsilon}} = \epsilon_t(\mathbf{u}_x\mathbf{u}_x + \mathbf{u}_y\mathbf{u}_y) + \epsilon_z\mathbf{u}_z\mathbf{u}_z, \tag{1}$$

$$\overline{\overline{\mu}} = \mu_t(\mathbf{u}_x\mathbf{u}_x + \mathbf{u}_y\mathbf{u}_y) + \mu_z\mathbf{u}_z\mathbf{u}_z, \tag{2}$$

any field can be decomposed in TE and TM parts with respect to the axial vector $\mathbf{u}_z$ as

$$\mathbf{E} = \mathbf{E}_{TE} + \mathbf{E}_{TM}, \quad \mathbf{H} = \mathbf{H}_{TE} + \mathbf{H}_{TM}, \tag{3}$$

satisfying

$$\mathbf{u}_z \cdot \mathbf{E}_{TE} = 0, \quad \mathbf{u}_z \cdot \mathbf{H}_{TM} = 0. \tag{4}$$

The decomposition can be simply demonstrated for a plane wave. In fact, because the fields of a plane wave in any medium satisfy $\mathbf{E}\cdot\mathbf{B} = 0$ and $\mathbf{H}\cdot\mathbf{D} = 0$, they also satisfy

$$\epsilon_t\mathbf{E}\cdot\mathbf{B} - \mu_t\mathbf{H}\cdot\mathbf{D} = (\epsilon_t\mu_z - \mu_t\epsilon_z)(\mathbf{u}_z\cdot\mathbf{E})(\mathbf{u}_z\cdot\mathbf{H}) = 0, \tag{5}$$

in the uniaxial medium (1), (2). Since any electromagnetic field outside the source region can be decomposed in a spectrum of plane waves, each component of which is either a TE wave or a TM wave, any field can be uniquely decomposed in TE and TM parts. For the special medium satisfying $\epsilon_t\mu_z - \mu_t\epsilon_z = 0$ the decomposition can be done but it is not unique.

In [1] the decomposition was generalized for bi-anisotropic media defined by medium dyadics of the form

$$\overline{\overline{\epsilon}} = \frac{1}{2\tau}(-\overline{\overline{\mathsf{B}}}^T + \mathbf{a}_2\mathbf{b}_1 + \mathbf{b}_2\mathbf{a}_1), \tag{6}$$

$$\overline{\overline{\xi}} = \mathbf{x}\times\overline{\overline{\mathsf{I}}} + \frac{1}{2\tau}(\mathbf{a}_2\mathbf{b}_2 + \mathbf{b}_2\mathbf{a}_2), \tag{7}$$

$$\overline{\overline{\zeta}} = \mathbf{z}\times\overline{\overline{\mathsf{I}}} + \frac{1}{2\eta}(\mathbf{a}_1\mathbf{b}_1 + \mathbf{b}_1\mathbf{a}_1), \tag{8}$$

978-1-4244-6051-9/11/$26.00

$$\overline{\overline{\mu}} = \frac{1}{2\eta}(\overline{\overline{\mathsf{B}}} + \mathbf{a}_1\mathbf{b}_2 + \mathbf{b}_1\mathbf{a}_2), \tag{9}$$

where $\mathbf{x}, \mathbf{z}$ are arbitrary vectors, τ, η are arbitrary scalars and $\overline{\overline{\mathsf{B}}}$ is an arbitrary dyadic. In this case the plane-wave fields satisfy

$$\eta\mathbf{E}\cdot\mathbf{B} + \tau\mathbf{H}\cdot\mathbf{B} = (\mathbf{a}_1\cdot\mathbf{E} + \mathbf{a}_2\cdot\mathbf{H})(\mathbf{b}_1\cdot\mathbf{E} + \mathbf{b}_2\cdot\mathbf{H}) = 0, \tag{10}$$

whence they can be decomposed to two parts satisfying either $\mathbf{a}_1\cdot\mathbf{E} + \mathbf{a}_2\cdot\mathbf{H} = 0$ or $\mathbf{b}_1\cdot\mathbf{E} + \mathbf{b}_2\cdot\mathbf{H} = 0$. Media defined by (6) − (9) were subsequently generalized by adding a scalar parameter [2] and they have been called decomposable media.

2 Four-dimensional representation

To be able to find more general classes of decomposable media let us consider the problem in terms of four-dimensional differential-form representation [3,4]. The Maxwell equations can be expressed as

$$\mathbf{d}\wedge\mathbf{\Phi} = 0, \qquad \mathbf{d}\wedge\mathbf{\Psi} = \boldsymbol{\gamma}, \tag{11}$$

with the electromagnetic field two-forms $\mathbf{\Phi} = \mathbf{B} + \mathbf{E}\wedge\boldsymbol{\varepsilon}_4$, $\mathbf{\Psi} = \mathbf{D} - \mathbf{H}\wedge\boldsymbol{\varepsilon}_4$ coupled by the medium equation

$$\mathbf{\Psi} = \overline{\overline{\mathsf{M}}}|\mathbf{\Phi}, \tag{12}$$

where the medium dyadic $\overline{\overline{\mathsf{M}}}$ mapping two-forms to two-forms corresponds to a 6×6 matrix. $\boldsymbol{\varepsilon}_4$ is a basis one-form corresponding to the temporal coordinate. It is often simpler to consider the modified medium dyadic $\overline{\overline{\mathsf{M}}}_g$ defined by $\overline{\overline{\mathsf{M}}}_g = \mathbf{e}_N\lfloor\overline{\overline{\mathsf{M}}}$, which maps two-forms to bivectors in terms of a quadrivector $\mathbf{e}_N = \mathbf{e}_{1234}$. Definitions and operational rules for differential forms, multivectors and dyadics applied in this paper can be found in [4]. Assuming plane-wave fields of the form $\mathbf{\Phi}(\mathbf{x}) = \mathbf{\Phi}\exp(\boldsymbol{\nu}|\mathbf{x})$, $\mathbf{\Psi}(\mathbf{x}) = \mathbf{\Psi}\exp(\boldsymbol{\nu}|\mathbf{x})$, from (11) the field two-forms can be expressed in terms of potential one-forms as

$$\mathbf{\Phi} = \boldsymbol{\nu}\wedge\boldsymbol{\phi}, \qquad \mathbf{\Psi} = \boldsymbol{\nu}\wedge\boldsymbol{\psi}, \tag{13}$$

whence they satisfy the three orthogonality conditions

$$\mathbf{\Phi}\cdot\mathbf{\Phi} = \mathbf{\Phi}|(\mathbf{e}_N\lfloor\overline{\overline{\mathsf{I}}}^{(2)T})|\mathbf{\Phi} = 0, \qquad \mathbf{\Phi}\cdot\mathbf{\Psi} = \mathbf{\Phi}|\overline{\overline{\mathsf{M}}}_g|\mathbf{\Phi} = 0, \qquad \mathbf{\Psi}\cdot\mathbf{\Psi} = \mathbf{\Phi}|\overline{\overline{\mathsf{M}}}_g^T\cdot\overline{\overline{\mathsf{M}}}_g|\mathbf{\Phi} = 0. \tag{14}$$

The dot product between two-forms is defined by $\mathbf{\Phi}\cdot\mathbf{\Psi} = \mathbf{e}_N|(\mathbf{\Phi}\wedge\mathbf{\Psi})$.

3 Decomposition condition

Let us require that the decomposition of fields in the 4D representation is based on orthogonality with respect to two chosen bivectors $\mathbf{A}, \mathbf{B}$ as

$$(\mathbf{A}|\mathbf{\Phi})(\mathbf{B}|\mathbf{\Phi}) = \mathbf{\Phi}|(\mathbf{AB})|\mathbf{\Phi} = 0. \tag{15}$$

Let us now define the class of decomposable media by requiring that for some coefficients α, β, γ the following condition is valid for any two-forms $\mathbf{\Phi}$,

$$\mathbf{\Phi}|(\alpha\mathbf{e}_N\lfloor\overline{\overline{\mathsf{I}}}^{(2)T} + 2\beta\overline{\overline{\mathsf{M}}}_g + \gamma\overline{\overline{\mathsf{M}}}_g^T\cdot\overline{\overline{\mathsf{M}}}_g)|\mathbf{\Phi} = 2\mathbf{\Phi}|(\mathbf{AB})|\mathbf{\Phi}, \tag{16}$$

the left-hand side of which vanishes due to (14). This leads to a symmetric dyadic equation for the medium dyadic $\overline{\overline{\mathsf{M}}}_g$,

$$\alpha\mathbf{e}_N\lfloor\overline{\overline{\mathsf{I}}}^{(2)T} + \beta(\overline{\overline{\mathsf{M}}}_g + \overline{\overline{\mathsf{M}}}_g^T) + \gamma\overline{\overline{\mathsf{M}}}_g^T\cdot\overline{\overline{\mathsf{M}}}_g = \mathbf{AB} + \mathbf{BA}. \tag{17}$$

Now the class of decomposable media (DCM) can be split in two parts as based on the scalar coefficient γ. In the case $\gamma = 0$, the second-order term in the decomposition condition (17) drops off and the subclass will be called that of special decomposable media (SDCM). For the proper DCM case we can normalize $\gamma = 1$ and the dyadic condition (17) is of the second order.

4 Solutions for the decomposable medium

The general solution of the first-order equation (17) with $\gamma = 0$ can be expressed in the form [5]

$$\overline{\overline{\mathsf{M}}}_g = \alpha \mathbf{e}_N \lfloor \overline{\overline{\mathsf{I}}}^{(2)T} + \mathbf{AB} + \overline{\overline{\mathsf{A}}}, \tag{18}$$

where $\overline{\overline{\mathsf{A}}}$ is any antisymmetric dyadic. Solution of the second-order equation (17) with $\gamma = 1$ can be based on solutions of the simplified dyadic equation

$$\overline{\overline{\mathsf{X}}}^T \cdot \overline{\overline{\mathsf{X}}} = \alpha \mathbf{e}_N \lfloor \overline{\overline{\mathsf{I}}}^{(2)T}, \tag{19}$$

which can be shown to possess two sets of solutions:

$$\overline{\overline{\mathsf{X}}} = A \mathbf{e}_N \lfloor \overline{\overline{\mathsf{I}}}^{(2)T} + B \overline{\overline{\mathsf{Q}}}^{(2)}, \tag{20}$$

$$\overline{\overline{\mathsf{X}}} = A \mathbf{e}_N \lfloor \overline{\overline{\mathsf{I}}}^{(2)T} + B \mathbf{e}_N \lfloor \overline{\overline{\mathsf{P}}}^{(2)}, \tag{21}$$

where the dyadic $\overline{\overline{\mathsf{Q}}}$ maps one-forms to vectors and the dyadic $\overline{\overline{\mathsf{P}}}$ maps one-forms to one-forms. These two sets of solutions give rise to two subclasses of decomposable media denoted here by QDCM and PDCM with the respective forms for the medium dyadics

$$\overline{\overline{\mathsf{M}}}_g = \alpha \mathbf{e}_N \lfloor \overline{\overline{\mathsf{I}}}^{(2)T} + M \overline{\overline{\mathsf{Q}}}^{(2)} + \mathbf{DA}, \tag{22}$$

$$\overline{\overline{\mathsf{M}}}_g = \alpha \mathbf{e}_N \lfloor \overline{\overline{\mathsf{I}}}^{(2)T} + M \mathbf{e}_N \lfloor \overline{\overline{\mathsf{P}}}^{(2)} + \mathbf{DA}. \tag{23}$$

The relation of the bivector $\mathbf{D}$ to the bivector $\mathbf{B}$ is different in these two cases and can be expressed as

$$\mathbf{B} = M \overline{\overline{\mathsf{Q}}}^{(2)T} \cdot \mathbf{D} + \frac{1}{2}(\mathbf{D} \cdot \mathbf{D})\mathbf{A}, \tag{24}$$

$$\mathbf{B} = M \overline{\overline{\mathsf{P}}}^{(2)T} | \mathbf{D} + \frac{1}{2}(\mathbf{D} \cdot \mathbf{D})\mathbf{A}. \tag{25}$$

One can solve $\mathbf{D}$ from these equations for given bivectors $\mathbf{A}, \mathbf{B}$ and dyadics $\overline{\overline{\mathsf{Q}}}, \overline{\overline{\mathsf{P}}}$. As a conclusion, there are three subclasses of decomposable media, denoted as QDCM, PDCM and SDCM.

5 Plane waves in decomposable media

For a plane wave the potential one-form satisfies the equation [4]

$$(\overline{\overline{\mathsf{M}}}_g \lfloor \lfloor \boldsymbol{\nu\nu}) | \boldsymbol{\phi} = 0. \tag{26}$$

which has non-trivial solutions $\boldsymbol{\phi} \neq 0$ only when the wave one-form $\boldsymbol{\nu}$ satisfies certain dispersion equation $D(\boldsymbol{\nu}) = 0$, which is of the fourth order in $\boldsymbol{\nu}$ for the general medium. One may note that the first term (axion term [3]) in (18), (22) and (23) does not contribute to (26) and can be omitted.

In the case of QDCM, (26) equals that of the generalized Q-medium studied previously [6], whence the dispersion equation can be obtained similarly. It splits in in two quadratic equations as

$$\boldsymbol{\nu} | \overline{\overline{\mathsf{Q}}} | \boldsymbol{\nu} = 0, \tag{27}$$

$$\boldsymbol{\nu} | (M \overline{\overline{\mathsf{Q}}} + \mathbf{AD} \lfloor \lfloor \overline{\overline{\mathsf{Q}}}^{-1}) | \boldsymbol{\nu} = 0. \tag{28}$$

One can show that (27) corresponds to $\mathbf{A} | \boldsymbol{\Phi} = 0$ and (28) to $\mathbf{B} | \boldsymbol{\Phi} = 0$.

In the case of PDCM, (26) coincides with that of the generalized P-medium whose dispersion equation was derived in [7]. In this case the dispersion equation can be split in two quadratic equations

$$\boldsymbol{\nu}|(\mathbf{D}\lfloor\overline{\overline{\mathsf{P}}})|\boldsymbol{\nu} = 0, \tag{29}$$

$$\boldsymbol{\nu}|(\mathbf{A}\lfloor\overline{\overline{\mathsf{P}}}^{-1})|\boldsymbol{\nu} = 0. \tag{30}$$

One can show that in this case (30) corresponds to $\mathbf{A}|\boldsymbol{\Phi} = 0$ and (30) to $\mathbf{B}|\boldsymbol{\Phi} = 0$.

Finally, in the case of SDCM when the antisymmetric dyadic $\overline{\overline{\mathsf{A}}}$ in (18) is expressed in terms of a trace-free dyadic $\overline{\overline{\mathsf{B}}}_o$ as

$$\overline{\overline{\mathsf{A}}} = (\overline{\overline{\mathsf{B}}}_o{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{I}}})^T, \qquad \mathrm{tr}\overline{\overline{\mathsf{B}}}_o = 0, \tag{31}$$

the fourth-order dispersion equation splits in two second-order equations

$$\boldsymbol{\nu}|(\overline{\overline{\mathsf{B}}}_o\rfloor\mathbf{A})|\boldsymbol{\nu} = 0, \qquad \boldsymbol{\nu}|(\overline{\overline{\mathsf{B}}}_o\rfloor\mathbf{B})|\boldsymbol{\nu} = 0, \tag{32}$$

the former of which corresponds to $\mathbf{A}|\boldsymbol{\Phi} = 0$ and, the latter, to $\mathbf{B}|\boldsymbol{\Phi} = 0$.

6 Conclusion

It was shown that, by starting from a four-dimensional representation of electromagnetic fields, the class of media in which fields can be decomposed in two sets in terms of polarization has been dramatically widened from that based on three-dimensional definition. The class of decomposable media can be naturally divided in three subclasses dubbed as QDCM, PDCM and SDCM. For each of the subclasses the dispersion equation associated with a plane wave was shown to split in two quadratic dispersion equations corresponding to the two polarizations of the decomposed fields.

7 References

1. I. V. Lindell, and F. Olyslager, “Generalized decomposition of electromagnetic fields in bi-anisotropic media,” *IEEE Trans. Antennas Propag.*, vol. 46, 1998, pp. 1584-1585.

2. F. Olyslager, and I. V. Lindell, “Field decomposition and factorization of the Helmholtz determinant operator for bianisotropic media,” *IEEE Trans. Antennas Propag.*, vol. 49, 2001, pp. 660-665.

3. F. W. Hehl, and Yu. N. Obukhov, *Foundations of Classical Electrodynamics*, Boston, Birkhäuser, 2003.

4. I. V. Lindell, *Differential Forms in Electromagnetics*, New York, Wiley 2004.

5. I. V. Lindell, L. Bergamin, and A. Favaro, “Decomposable medium conditions in four-dimensional representation”, *Trans. IEEE Antennas Propag.*, submitted.

6. I. V. Lindell, and H. Wallén, “Generalized Q-media and field decomposition in differential-form approach,” *J. Electro. Waves Appl.*, vol. 18, no.8, 2004, pp. 1045-1056.

7. I. V. Lindell, L. Bergamin, and A. Favaro, “The class of electromagnetic P-media and its generalization,” *Prog. Electromag. Res. B*, vol. 28, 2011, pp. 143-162.