Physical bounds on scattering by metamaterials

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Abstract

The extinction cross section, which quantifies the scattering and absorption of electromagnetic waves, is a function of the frequency of the external excitation. Due to causality of the scattered field in the forward direction and the optical theorem, the extinction cross section satisfies a sum rule. The result states that the weighted integral of the extinction cross section over all frequencies is related to the static polarizability dyadics of the object, irrespectively whether the permittivity or the permeability attain negative real parts at higher frequencies or not. From this sum rule, physical bounds on scattering are derived for general material models. The theory is illustrated with closed form expressions of the polarizability dyadics of both a temporally dispersive sphere and a chiral sphere, and compared with the outcome of some explicit numerical examples.

1 Theory — a scattering identity for general material models

Recently, several intriguing identities concerning scattering of electromagnetic waves by finite objects have been developed [4–6]. The identities are examples of sum rules, which are well established tools in *e.g.*, quantum scattering [3], but seem to be less known and exploited in electrodynamics and acoustics. A bewildering consequence of these sum rules is that the static properties of an scatterer (its polarizability dyadics) determine the weighted integral of the extinction cross section over all frequencies. The assumptions behind the sum rules are linearity, causality, energy conservation, and invariance under time translations. Due to these very general assumptions, the results are shown useful for a large class of scatterers — not just simple canonical geometries. In addition to being instrumental in understanding the broadband scattering properties of an object, these sum rules also give several new interesting results in antenna applications [1].

At a single frequency, when causality has no meaning, the material modeling of the scatterer is less critical. However, dealing with the broadband properties of an scatterer, it becomes important to use physically suitable dispersion models. The isotropic material models are well known and used for more than a century. In the 90's, modeling of optical activity led to the introduction of chiral models (bi-isotropic materials), and recently much effort has focused on understanding the effects of negative real parts of the material parameters (metamaterials). In this paper we exploit a specific sum rule, which restricts the broadband scattering properties of an object. The sum rule depends only on the well defined static properties of the object, which are conveniently tractable by numerical computations. As a consequence, it does not matter whether the object shows metamaterial properties at higher frequencies or not, its weighted integral of the extinction cross section stays the same.

Consider the direct scattering problem of a plane electromagnetic wave $\hat{\boldsymbol{e}}\mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}\cdot\boldsymbol{x}}$ (time dependence $\mathrm{e}^{-\mathrm{i}\omega t}$) of unit amplitude impinging in the $\hat{\boldsymbol{k}}$-direction on a target embedded in free space. The target can be a single scatterer (homogeneous or not) or it may consist of several parts. The material of the scatterer is modeled by a set of linear and passive constitutive relations which are assumed to be invariant under time translations (*i.e.*, stationary constitutive relations). The scattering dyadic (amplitude) $\mathbf{S}$ is defined in terms of the scattered electric field $\boldsymbol{E}_\mathrm{s}$ and the polarization of the incident field $\hat{\boldsymbol{e}}$ as [3]

$$\mathbf{S}(k;\hat{\boldsymbol{k}}\curvearrowright\hat{\boldsymbol{x}})\cdot\hat{\boldsymbol{e}}=\lim_{x\to\infty}x\mathrm{e}^{-\mathrm{i}kx}\boldsymbol{E}_\mathrm{s}(k;\boldsymbol{x})$$

where $x=|\boldsymbol{x}|$ denotes the magnitude of the position vector, and $\hat{\boldsymbol{x}}=\boldsymbol{x}/x$. A target's overall scattering properties are commonly quantified by the scattering cross section σ_s, defined as the total power scattered in all directions divided by the incident power flux. The extinction cross section $\sigma_\mathrm{ext}=\sigma_\mathrm{s}+\sigma_\mathrm{a}$ is defined as the sum of the scattering and absorption cross sections, where the latter is a measure of the absorbed power in the target. The extinction cross section is also determined from the scattering dyadic in the forward direction, $\hat{\boldsymbol{x}} = \hat{\boldsymbol{k}}$, *viz.*,

$$\sigma_{\rm ext}(k;\hat{\boldsymbol{k}},\hat{\boldsymbol{e}}) = \frac{4\pi}{k}\,\mathrm{Im}\left\{\hat{\boldsymbol{e}}^*\cdot\mathbf{S}(k;\hat{\boldsymbol{k}}\curvearrowright\hat{\boldsymbol{k}})\cdot\hat{\boldsymbol{e}}\right\} \tag{1}$$

where an asterisk denotes the complex conjugate. Relation (1) is known as the optical theorem, and it is applicable to many different wave phenomena such as acoustic waves, electromagnetic waves, and elementary particles [3].

A sum rule for the combined effect of scattering and absorption of electromagnetic waves is derived in Ref. 4 from the holomorphic properties of the forward scattering dyadic. The result is valid for any scatterers satisfying the general assumptions made above. The sum rule, known as the integrated extinction, reads [4]

$$\int_0^\infty \frac{\sigma_{\rm ext}(k)}{k^2}\,\mathrm{d}k = \frac{\pi}{2}\left(\hat{\boldsymbol{e}}^*\cdot\boldsymbol{\gamma}_{\rm e}\cdot\hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}}\times\hat{\boldsymbol{e}}^*)\cdot\boldsymbol{\gamma}_{\rm m}\cdot(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{e}})\right) \tag{2}$$

Here, $\boldsymbol{\gamma}_{\rm e}$ and $\boldsymbol{\gamma}_{\rm m}$ denote the electric and magnetic polarizability dyadics, respectively [4]. The effects of (2) for metamaterials are exploited in this paper, and *e.g.*, we immediately observe that all materials with the same static properties have identical integrated extinction.

The integrand on the left-hand side of (2) is non-negative. Therefore, for any finite frequency interval $K = k_0[1 - B/2, 1 + B/2]$ with center frequency k_0 and relative bandwidth B, the identity implies

$$\frac{B\sigma_{\rm ext}(\kappa)}{k_0(1-B^2/4)} = \int_K \frac{\sigma_{\rm ext}(k)}{k^2}\,\mathrm{d}k \le \frac{\pi}{2}\left(\hat{\boldsymbol{e}}^*\cdot\boldsymbol{\gamma}_{\rm e}\cdot\hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}}\times\hat{\boldsymbol{e}}^*)\cdot\boldsymbol{\gamma}_{\rm m}\cdot(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{e}})\right) \tag{3}$$

for some $\kappa \in K$. For all scatterers with the same static polarizability dyadics, this inequality shows that large scattering in a frequency interval is traded for smaller bandwidth, since the left-hand side of the inequality is bounded from above by the right-hand side. Corresponding inequalities can be obtained for the scattering and absorption cross sections, $\sigma_{\rm s}$ and $\sigma_{\rm a}$, respectively.

2 Theory — material modeling with applications to chiral and dispersive spheres

In this section, we illustrate the sum rule, (2), with two examples. The material of the scatterer is modeled by the following reciprocal, bi-isotropic constitutive relations [2]

$$\begin{cases} \boldsymbol{D}(\omega,\boldsymbol{r}) = \epsilon_0\left\{\epsilon(\omega)\boldsymbol{E}(\omega,\boldsymbol{r}) + \mathrm{i}\eta_0\chi(\omega)\cdot\boldsymbol{H}(\omega,\boldsymbol{r})\right\} \\ c_0\boldsymbol{B}(\omega,\boldsymbol{r}) = -\mathrm{i}\chi(\omega)\boldsymbol{E}(\omega,\boldsymbol{r}) + \eta_0\mu(\omega)\boldsymbol{H}(\omega,\boldsymbol{r}) \end{cases}$$

where $\eta_0 = \sqrt{\mu_0/\epsilon_0}$ denotes the wave impedance of vacuum, and ϵ_0, μ_0, and c_0 are the permittivity, permeability and speed of light of vacuum, respectively. These constitutive relations contain three different material parameters, *viz.*, the permittivity $\epsilon(\omega)$, the permeability $\mu(\omega)$, and the chirality $\chi(\omega)$. Modeling realistic materials, these parameters have to satisfy additional requirements. In particular, if the object is passive, the following conditions have to be fulfilled:

$$\mathrm{Im}\,\epsilon(\omega) \ge 0, \qquad \mathrm{Im}\,\mu(\omega) \ge 0, \qquad \mathrm{Im}\,\epsilon(\omega)\,\mathrm{Im}\,\mu(\omega) \ge \left(\mathrm{Im}\,\chi(\omega)\right)^2, \quad \text{for all } \omega > 0 \tag{4}$$

The first two inequalities are identical to the requirements of a passive isotropic material, and the third one is added for a bi-isotropic material. We observe that the imaginary part of the chirality cannot be chosen arbitrary, *viz.*, a large imaginary part of the chirality in general creates an active material.

In order to use the identity in (2), we have to know the polarizability dyadics $\boldsymbol{\gamma}_{\rm e}$ and $\boldsymbol{\gamma}_{\rm m}$. For a chiral sphere of radius a embedded in free space these are known [2], *viz.*,

$$\begin{cases} \boldsymbol{\gamma}_{\rm e} = \dfrac{4\pi a^3\left((\epsilon(0)-1)(\mu(0)+2)-\chi^2(0)\right)}{(\epsilon(0)+2)(\mu(0)+2)-\chi^2(0)}\mathbf{I} - \dfrac{12\pi\chi(0)a^3}{(\epsilon(0)+2)(\mu(0)+2)-\chi^2(0)}\hat{\boldsymbol{k}}\times\mathbf{I} \\ \boldsymbol{\gamma}_{\rm m} = \dfrac{4\pi a^3\left((\epsilon(0)+2)(\mu(0)-1)-\chi^2(0)\right)}{(\epsilon(0)+2)(\mu(0)+2)-\chi^2(0)}\mathbf{I} + \dfrac{12\pi\chi(0)a^3}{(\epsilon(0)+2)(\mu(0)+2)-\chi^2(0)}\hat{\boldsymbol{k}}\times\mathbf{I} \end{cases}$$

where $\mathbf{I}$ denotes the unit dyadic in three dimensions. The right hand side of (2) then becomes

$$\hat{\boldsymbol{e}}^* \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}^*) \cdot \boldsymbol{\gamma}_{\mathrm{m}} \cdot (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}) = 4\pi a^3 \frac{(\epsilon(0)-1)(\mu(0)+2) + (\epsilon(0)+2)(\mu(0)-1) - 2\chi^2(0)}{(\epsilon(0)+2)(\mu(0)+2) - \chi^2(0)}$$

This is the general expression containing the polarizability dyadics for a chiral sphere of radius a with static values of the permittivity and the permeability, $\epsilon(0)$ and $\mu(0)$, and the chirality $\chi(0)$, respectively. Under normal conditions, the chiral effects disappear in the static limit, leading to $\chi(0) = 0$. For this non-chiral static limit, the right hand side is the sum of the ordinary electric and magnetic polarizability contributions.

For the dispersive material model, we adopt a Lorentz model, *viz.*,

$$\epsilon(\omega) = \epsilon_\infty + \sum_{n=1}^{2} f(\omega, \omega_{\mathrm{pe}n}, \omega_{0\mathrm{e}n}, \nu_{\mathrm{e}n}), \qquad \mu(\omega) = \mu_\infty + \sum_{n=1}^{2} f(\omega, \omega_{\mathrm{pm}n}, \omega_{0\mathrm{m}n}, \nu_{\mathrm{m}n}) \tag{5}$$

where

$$f(\omega, \omega_{\mathrm{p}}, \omega_0, \nu) = -\frac{\omega_{\mathrm{p}}^2}{\omega^2 - \omega_0^2 + \mathrm{i}\omega\nu}$$

The two positive constants ϵ_∞ and μ_∞ are the optical responses of the permittivity and the permeability, respectively, and the positive constants $\omega_{\mathrm{pe}n}$ and $\omega_{\mathrm{pm}n}$, $n = 1, 2$, are the plasma frequencies that model the strength of the dispersion. The resonance frequencies of the models are determined by the angular frequencies, $\omega_{0\mathrm{e}n}$ and $\omega_{0\mathrm{m}n}$, $n = 1, 2$, and the collision frequencies $\nu_{\mathrm{e}n}$ and $\nu_{\mathrm{m}n}$, The explicit values of these quantities in the static limit ($\omega = 0$) are

$$\epsilon(0) = \epsilon_\infty + \frac{\omega_{\mathrm{pe}}^2}{\omega_{0\mathrm{e}}^2}, \qquad \mu(0) = \mu_\infty + \frac{\omega_{\mathrm{pm}}^2}{\omega_{0\mathrm{m}}^2}, \qquad \chi(0) = 0$$

which with $\epsilon_\infty = \mu_\infty = 1$ lead to a simplified right-hand side of (2), *i.e.*,

$$\frac{\pi}{2}\left(\hat{\boldsymbol{e}}^* \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}^*) \cdot \boldsymbol{\gamma}_{\mathrm{m}} \cdot (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}})\right) = 2\pi^2 a^3 \left(\frac{\omega_{\mathrm{pe}}^2}{\omega_{\mathrm{pe}}^2 + 3\omega_{0\mathrm{e}}^2} + \frac{\omega_{\mathrm{pm}}^2}{\omega_{\mathrm{pm}}^2 + 3\omega_{0\mathrm{m}}^2}\right)$$

independently of the parameters ω_{c} and $\omega_{0\mathrm{c}}$ and the collision frequencies ν_{e}, ν_{m}, and ν_{c}.

3 Two numerical examples with a dispersive and a chiral sphere

We illustrate the theory in Sections 2 and 3 with two numerical examples. The first illustrates scattering by a Lorentz dispersive sphere of radius a. The following material parameters are used:

$$\epsilon_\infty = \mu_\infty = 1, \quad \begin{cases} \omega_{\mathrm{pe1}} = \omega_{\mathrm{pm1}} = 5\pi \cdot 10^9\ \mathrm{rad/s} \\ \omega_{\mathrm{pe2}} = \omega_{\mathrm{pm2}} = 0,8\omega_{\mathrm{pe1}} \\ \omega_{0\mathrm{e1}} = \omega_{0\mathrm{m1}} = 50\pi \cdot 10^9\ \mathrm{rad/s} \\ \omega_{0\mathrm{e2}} = \omega_{0\mathrm{m2}} = 0.5\omega_{0\mathrm{e1}} \\ \nu_{\mathrm{e1}} = \nu_{\mathrm{m1}} = \nu_{\mathrm{e2}} = \nu_{\mathrm{m2}} = 2 \cdot 10^9\ \mathrm{Hz} \end{cases} \text{(I)}, \quad \begin{cases} \omega_{\mathrm{pe1}} = \omega_{\mathrm{pm1}} = 5\pi \cdot 10^9\ \mathrm{rad/s} \\ \omega_{\mathrm{pe2}} = \omega_{\mathrm{pm2}} = 0,4\omega_{\mathrm{pe1}} \\ \omega_{0\mathrm{e1}} = \omega_{0\mathrm{m1}} = 50\pi \cdot 10^9\ \mathrm{rad/s} \\ \omega_{0\mathrm{e2}} = \omega_{0\mathrm{m2}} = 0.25\omega_{0\mathrm{e1}} \\ \nu_{\mathrm{e1}} = \nu_{\mathrm{m1}} = \nu_{\mathrm{e2}} = \nu_{\mathrm{m2}} = 2 \cdot 10^9\ \mathrm{Hz} \end{cases} \text{(II)} \tag{6}$$

These explicit values of the parameters give both negative real parts of ϵ and μ in the vicinity of the resonances, *i.e.*, in these frequency intervals it is a model of a metamaterial. The extinction cross section is depicted in the left part of Figure 1. The general effect that is clearly illustrated is that if the lower resonances is divided by a factor of two, the bandwidth and the height is decreased with a factor of two, exactly what is predicted by the sum rule in (3). This feature is demonstrated by the the box confinements.

In the second example we illustrate the sum rule with scattering by a non-dispersive, chiral sphere. For physical reasons, the chirality must be zero at zero frequency. However, any non-physical value of $\chi(0)$, the identity in (2) still serves as an excellent test of the numerical performance of the code. In Figure 1 we depict two different scatterers that have the same integrated extinction in (2). Data is given by the constant values

$$\epsilon = 4.0, \quad \mu = 1, \quad \chi = 0, \quad \text{non-chiral case}; \qquad \epsilon = 4.04, \quad \mu = 1, \quad \chi = 0.2, \quad \text{chiral case} \tag{7}$$

We see that no dramatic change in the extinction cross section occurs when chirality is introduced.

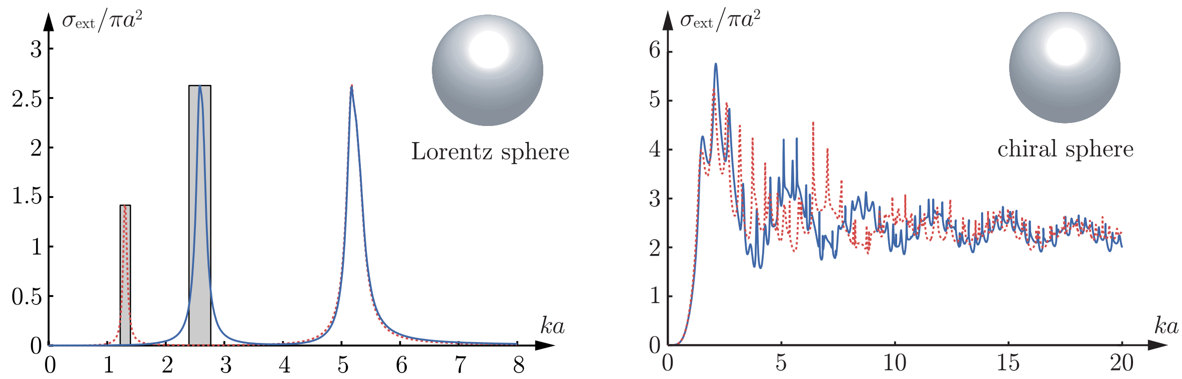

Figure 1: **Left:** The extinction cross section $\sigma_{\rm ext}$ (scaled with πa^2) for a isotropic sphere of radius $a = 1\,\mathrm{cm}$ as a function of ka with data in (6). The solid (blue) and the dotted (red) curves show the cross section for two different resonance models with the same integrated extinction. The weighted area under all curves has the same value (0.1474). **Right:** The extinction cross section $\sigma_{\rm ext}$ (scaled with πa^2) as a function of ka with data in (7). The solid (blue) curve is the cross section without chirality, and the broken (red) curve with chirality ($\chi = 0.2$). The weighted area under both curves has the same value (π).

4 Conclusions

This paper exploits a sum rule of the extinction cross section to find bounds on scattering of electromagnetic waves by an object. The theory is illustrated by scattering by both a Lorentz dispersive sphere and a chiral sphere. The underlying dispersion model has simultaneously negative real parts of the permittivity and the permeability in certain frequency intervals, *i.e.*, a metamaterial. The integrated extinction, which exclusively is determined by the static properties of the object, limits the scattering properties of the object. Specifically, it is found that large scattering effects always have to be compensated by a loss of bandwidth. This loss of bandwidth can be quantified.

References

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