# ANALYSIS OF ELECTROMAGNETIC FIELD IN INHOMOGENEOUS MEDIA BY FOURIER SERIES EXPNSION METHODS - THE CASE OF A DIELECTRIC CONSTANT MIXED IN POSITIVE AND NEGATIVE REGION- 

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## 1. Introduction

Recently, the scattering and guiding problems of the inhomogeneous media have been of considerable interest, such as optical fiber gratings , photonic bandgap crystals, frequency selective devices, and a negative media[1].In the negative megium,such as a plasma or a metallic grating, the permittivity has both positive and negative regions. Yamaguch et al. proposed to the modified multilayer approximation method (MMA) in the positive and negative regions by linear profiles to the oblique angle incidences of the TM wave[2]. However, when the permittivity has a positive and touches zero, MMA cannot be applicable. On the other hand, there cannot be also applicable to the electromagnetic field in the inhomogeneous layer though the homogeneous multilayer approximation method (HMA) using extrapolation method adding the loss term.

In this paper, we proposed a new method for the electromagnetic fields with inhomogeneous medium mixed the positive and negative region by the combination of improved Fourier series expansion method[3] using the extrapolation

method[4].
Numerical results are given for the reflection and transmission coefficients ,and the electromagnetic fields in the positive and negative regions including the case of when the permittivity profiles touches zero for the case of TM wave using the extrapolation method which obtains the correct value of the eigenvalue and eigenvectors .

## 2. Method of Analysis

We consider inhomogeneous medium mixed in positive/negative as shown in Fig.1(a). The structure is uniform in the $y$-direction and the permittivity $\varepsilon_{2}(z)$ including singular points at $z=z_{1}$ and $z=z_{2}$ (see Fig.1(b)). The permeability is assumed to be $\mu_{0}$. The time dependence is $\exp (-i \omega t)$ and suppressed throughout. In the formulation, the TM wave is discussed. When the TM wave (the magnetic field has only the $y$ component) is assumed to be incident from $z<0$ at the angle $\theta_{0}$, the magnetic fields in the regions $S_{1}(z \leq 0)$ and $S_{3}(z \geq d)$ are expressed[3] as

Fig. 1 Structure of the inhomogeneous media with a dielectric constant mixed a positive and negative regions. (a) Coordinate system, (b) Approximated inhomogeneous layers.

$$
\begin{aligned}
& \frac{S_{1}(z \leq 0):}{H_{y}^{(1)}=e^{i k_{1}\left(x \sin \theta_{0}+z \cos \theta_{0}\right)}+R e^{i k_{1}\left(x \sin \theta_{0}-z \cos \theta_{0}\right)}} \\
& k_{1} \triangleq \omega \sqrt{\varepsilon_{1} \mu_{0}}=k_{0} \sqrt{\varepsilon_{1} / \varepsilon_{0}} \\
& \frac{S_{3}(z \geq d):}{H_{y}^{(3)}=T e^{i\left[k_{1} x \sin \theta_{0}+k_{3 z}(z-d)\right\}}} \\
& k_{3 x} \triangleq \sqrt{k_{3}{ }^{2}-\left(k_{1} \sin \theta_{0}\right)^{2}} ; k_{0} \triangleq 2 \pi / \lambda, j=1,3 .
\end{aligned}
$$

where $\lambda$ is the wavelength in free space, $R$, and $T$ are the reflection and transmission coefficients to be determined by boundary conditions.
The inhomogeneous layer $(0<z<d)$ consists periodically stratified layers which is the iteration of the permittivity $\varepsilon_{d}(z)\left[=\varepsilon_{2}(z)\right]$ in the original region $(0<z<d)$ [3]. The modal component of magnetic field can be written as $H_{y}^{(2)}=H(z) e^{i k_{1} x \sin \theta_{0}}$, and $H(z)$ must satisfy the following wave equation:

$$
\begin{align*}
\frac{d^{2} H(z)}{d z^{2}} & -\frac{1}{\varepsilon_{d}(z)} \frac{d \varepsilon_{d}(z)}{d z} \frac{d H(z)}{d z}  \tag{3}\\
& +\left[k_{0}^{2} \varepsilon_{d}(z) / \varepsilon_{0}-\left(k_{1} \sin \theta_{0}\right)^{2}\right] H(z)=0
\end{align*}
$$

Taking into account the Floquet's theorem , $H(z)$ can be approximated by the finite Fourier series as

$$
\begin{equation*}
H(z)=e^{i h z} \sum_{n=-N}^{N} u_{n} e^{i 2 \pi n z / d} \tag{4}
\end{equation*}
$$

To the obtain the correct solution in the analysis, $\varepsilon_{d}(z)$ is adding the expressed loss term $\sigma^{[4]}$

$$
\begin{equation*}
\tilde{\varepsilon}_{d}(z) / \varepsilon_{0} \triangleq \varepsilon_{d}(z) / \varepsilon_{0}+i \sigma(\sigma \geq 0) \tag{5}
\end{equation*}
$$

Substituting Eq.(5) and (4) into Eq.(3) and multiplying both side by $\tilde{\varepsilon}_{d}(z) e^{-i 2 \pi m z / d}$, and rearranging after integrating with respect to $z$ in the interval $0<z<d$.We get the following
equation in regard to $h$ [3].

$$
\begin{equation*}
h^{2} \mathbf{M} \mathbf{U}+h \mathbf{C U}+\mathbf{K} \mathbf{U}=0 \tag{6}
\end{equation*}
$$

where,

$$
\begin{align*}
& \quad \mathbf{U}^{(l)} \triangleq\left[u_{-N}, \cdots, u_{0}, \cdots u_{N}\right]^{T} T: \text { transpose }, \\
& \mathbf{M} \triangleq\left[\eta_{m, n}\right], \quad \mathbf{C} \triangleq\left[\zeta_{m, n}\right], \mathbf{K} \triangleq\left[\gamma_{m, n}\right] \\
& \left.\zeta_{n, m} \triangleq \frac{2 \pi}{d}\{2 n+(n-m)\} \eta_{n, m}\right\} \\
& \gamma_{n, m} \triangleq\left[\left(\frac{2 \pi}{d}\right)^{2}\left(n^{2}+n(n-m)\right)+\left(k_{0} \sin \theta_{0}\right)^{2}\right] \eta_{n, m}-\xi_{n, m} \\
& \quad m, n=(-N, \cdots, 0, \cdots, N) \tag{7}
\end{align*}
$$

$$
\begin{aligned}
& \eta_{n, m} \triangleq \frac{1}{d} \int_{0}^{p}\left\{\tilde{\varepsilon}(z) / \varepsilon_{0}\right\} e^{i 2 \pi(n-m) z / p} d z \\
& \xi_{n, m} \triangleq \frac{k_{0}^{2}}{d} \int_{0}^{d}\left\{\tilde{\varepsilon}_{d}(z) / \varepsilon_{0}\right\}^{2} e^{i 2 \pi(n-m) z / p} d z
\end{aligned}
$$

Letting $\mathbf{v}=h \mathbf{U}$ and Equation.(6) is reduced to the following conventional eigenvalue equation

$$
\begin{equation*}
\mathbf{A W}=h \mathbf{W} \tag{8}
\end{equation*}
$$

$\mathbf{A} \triangleq\left[\begin{array}{cc}\mathbf{0} & \mathbf{1} \\ -\mathbf{M}^{-1} \mathbf{K} & -\mathbf{M}^{-1} \mathbf{C}\end{array}\right], \quad \mathbf{W} \triangleq\left[\begin{array}{l}\mathbf{U} \\ \mathbf{v}\end{array}\right]$,
where, $\mathbf{1}$ :unit vectore, $\mathbf{M}^{-1}$ :inverse matrix of $\mathbf{M}$.
When it get an eigenvalue $h_{0}(=\beta+i \alpha)$ obtained Equation.(8) for $N \rightarrow \infty,-h_{0}(=-\beta-i \alpha)$ is also the solution, and $\left( \pm h_{0} \pm n\right)$ are also solutions. Therefore we selected $h_{0}$ by convergence characteristics of $\left( \pm h_{0} \pm n\right)$.In the lossless case, we obtain the eigenvalue $h_{\mathrm{EV}}$ is according to the following extrapolation equation $\left(a_{0}=h_{\mathrm{EV}}\right)$ from the loss parameters for $\sigma_{j}(j=1 \sim 3){ }^{[5]}$.

$$
\begin{equation*}
h_{0}\left(\sigma_{j}\right)=a_{0}+a_{1} \cdot \sigma_{j}+a_{3} \cdot\left(\sigma_{j}\right)^{2} ; j=1 \sim 3 \tag{9}
\end{equation*}
$$

In the same manner, the eigenvectors $u_{n}^{(\mathrm{EV})}$ case are also obtained the following extrapolation equation $\left(b_{0}=u_{n}{ }^{(\mathbb{E V V})}\right)$ from the loss parameters for $\sigma_{j}(j=1 \sim 3)$ [5].
$u_{n}{ }^{(E V)}\left(\sigma_{j}\right)=b_{0}+b_{1} \cdot \sigma_{j}+b_{3} \cdot\left(\sigma_{j}\right)^{2} ; j=1 \sim 3$
$\frac{S_{2}(0<z<d):}{\text { Using } h_{E V} \text { and }}-h_{E V}$, and the corresponding eigenvectors $u_{n}{ }^{(\mathrm{EV})}$ and $u_{n}{ }^{(-\mathrm{EV})}$, the electromagnetic fields are expanded appropriately by a finite Fourier series[4].

$$
\begin{align*}
& H_{y}^{(2)}=e^{i k_{1} \sin \theta_{0} x}\left[t_{v}^{(2)} e^{i h_{E V} z} f_{E V}(z)\right.  \tag{11}\\
& \left.\quad+r_{v}^{(2)} e^{-i k_{(-E V)} z} f_{(-E N)}(z)\right] \\
& E_{x}^{(2)}=\{-i \omega \varepsilon(z)\}^{-1} \partial H_{y}^{(2)} / \partial z  \tag{12}\\
& f_{E V}(z) \triangleq \sum_{n=-N}^{N} u_{n}^{(E V)} e^{i \frac{i \pi n}{d} z}, f_{(-E V)}(z) \triangleq \sum_{n=-N}^{N} u_{n}^{(-E V)} e^{i \frac{i \pi n}{d} z}
\end{align*}
$$

where $t_{v}^{(2)}$ and $r_{v}^{(2)}$ are unknown coefficients from the boundary conditions.
Using the boundary conditions , $R\left(=R_{E V}\right)$, and $T\left(=T_{E V}\right)$ are obtains following equation ${ }^{[4]}$.
$R_{E V}(N)=T_{E V}(N)-1$
$T_{E V}(N)=2\left[f_{e v}(d) D-f_{(-e v)}(d) C\right] /(A D-C D)$


Fig. $2|\beta d / 2 \pi|$ and $|\alpha d / 2 \pi|$.vs. $n$ for $\sigma=0.01$ in the case of the sinusoidal profile.


Fig. $3\left|R_{E V}(N)\right|^{2},\left|T_{E V}(N)\right|^{2}$ and G.vs. $\theta_{0}$


Fig. 4 Electromagnetic-Field in the case of sinusoidal profile


Fig. 5 Electromagnetic-field in the case of $\varepsilon_{2}(z) \geq 0$.

Where,
$A \triangleq f_{E V}(0)+\frac{\varepsilon_{1} F_{1}}{\varepsilon_{d}(0) k_{1} \cos \theta_{0}}, \quad B \triangleq f_{(-E V)}(0)+\frac{\varepsilon_{1} F_{2}}{\varepsilon_{d}(0) k_{1} \cos \theta_{0}}$
$C \triangleq f_{E V}(d)-\frac{\varepsilon_{3} F_{1}}{\varepsilon_{d}(d) k_{3 x}} e^{i k_{E V} d}, \quad D \triangleq f_{(-E V)}(d)-\frac{\varepsilon_{3} F_{2}}{\varepsilon_{d}(d) k_{3 x}} e^{-i k_{(-E V)} d}$
$F_{1} \triangleq \sum_{n=-N}^{N}\left(\frac{2 \pi n}{d}+h_{E V}\right) u_{n}^{(E V)}, F_{2} \triangleq \sum_{n=-N}^{N}\left(\frac{2 \pi n}{d}-h_{E V}\right) u_{n}^{(-E V)}$

## 3. Numerical Analysis

We consider the following sinusoidal profile in inhomogeneous media such a plasma medium for the TM wave:

$$
\begin{gather*}
\varepsilon_{2}(z) / \varepsilon_{0}=\varepsilon_{A}\{1+\delta \cos (2 \pi z / d)\}  \tag{14}\\
\varepsilon_{A} \triangleq \frac{\left[\varepsilon_{2}(\max )+\varepsilon_{2}(\min )\right]}{2}, \delta \triangleq \frac{\left[\varepsilon_{2}(\max )-\varepsilon_{2}(\min )\right]}{\left[\varepsilon_{2}(\max )+\varepsilon_{2}(\min )\right]}
\end{gather*}
$$

It has two singularity at $z_{1} / d \cong 0.07$ and $z_{2} / d \cong 0.93$.The values of parameters chosen are $\varepsilon_{1}=\varepsilon_{3}=\varepsilon_{0}, \varepsilon_{2}(\max )=2, \varepsilon_{2}(\min )=-0.1$ and $\sqrt{A} \lambda / d=0.8$.
Figures 2 shows the convergence of propagation constants $|\beta d /(2 \pi)|$ and $|\alpha d /(2 \pi)|$ to normalized $h_{0}(=\beta+i \alpha)$ with $N=50$, and $\theta_{0}=30^{\circ}$. From in Figures 2, the number chosen is $\mathrm{n}(=2 N+1=101)$ gives good convergence with comparison of exact solution which obtained MMA.

Using this number $n(=2 N+1)$, Figures 3 shows the power reflection coefficient $\left|R_{E V}(N)\right|^{2}$,the power transmission coefficient $\left|T_{E V}(N)\right|^{2}$ and the power loss of energy difference $G\left[\triangleq 1-\left|R_{E V}(N)\right|^{2}-\left|T_{E V}(N)\right|^{2}\right]$ for various values of incident angle $\theta_{0}$ using extrapolation method to select the loss term at $\sigma_{1}=0.011$, $\sigma_{2}=0.01$ and,$\sigma_{3}=0.09$.The results of the present method are in good agreement with those of exact solutions.
Figures 4(a) and 4(b) show the normalized magnetic fields $\left|H_{y}{ }^{(2)} / H_{y}{ }^{(i)}\right|^{2}$ and the normalized electric fields $\left|E_{x}^{(2)} / E_{x}^{(i)}\right|^{2}$ for various values of $z / d$ at $\theta_{0}=30^{\circ}$ with the same parameters as in Fig.3. From in Figs.4, .the results of the present method are in good agreement with those of exact solutions. For the $\left|E_{x}^{(2)} / E_{x}^{(i)}\right|^{2}$, the effect of singular point is seen clearly at $z / d \cong 0,07$ and 0.93 , but it is finite value (zero) at this point because of limitation.

Figures 5(a) and 5(b) show the normalized magnetic fields $\left|H_{y}{ }^{(2)} / H_{y}{ }^{(i)}\right|^{2}$ and the normalized electric fields $\left|E_{x}^{(2)} / E_{x}^{(i)}\right|^{2}$ for various values of $z / d$ at $\theta_{0}=30^{\circ}, 45^{\circ}$, and $60^{\circ}$ with the same parameters as in Fig. 3 for the case of the permittivity was positive and contained the singular point (zero).From in Figs.5(a), the peaks of $\left|H_{y}{ }^{(2)} / H_{y}{ }^{(i)}\right|^{2}$ move toward smaller $z / d$ as $\theta_{0}$ increases. On the other hand, for $\left|E_{x}^{(2)} / E_{x}^{(i)}\right|^{2}$ case, the effect of singular point is seen clearly at $z / d \cong 0$ and 1 , but it is finite value(zero) at this point because of limitation.

## 4. Conclusions

In this paper, we proposed a new method for the electromagnetic fields with inhomogeneous medium mixed the positive and negative regions by the combination of improved Fourier series
expansion method using the extrapolation method.
Numerical results are given for the reflection and transmission coefficients , and the electromagnetic fields in the positive and negative regions including the case of when the permittivity profiles touches zero for the case of TM wave using the extrapolation method which obtains the correct value of the eigenvalue and eigenvectors.
The results of our method are in good agreement with exact solusion which is obtained by MMA.

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