TENSORIAL FORMALISM APPLIED TO PROPAGATION PROBLEM

Aniss Hasnaâ, Gilles Y. Delisle***
Laboratory of Research in Underground Communication, Val d'Or, Québec, Canada
**International Institute of Telecommunications, Montréal, Québec, Canada*

INTRODUCTION

For structures such as feed guides of RF equipment, optical waveguides or TEM /GTEM chambers, it is necessary to numerically simulate their behaviour at different frequencies and thus determine the optimal functioning dimensions to reduce manufacture and test costs. It is very important to have a simulation technique that will yield fast and accurate results.

Our approach presents a new way of using tensorial formalism. Up until now, this formalism was always developed by considering only the covariant components E and H [1]. However, in this paper, the selection of the couple (E, B) for the study of the electromagnetic behaviour of the structure is justified. In the first section, the tensorial formalism is exposed and the concept of covariance and contravariance is defined. In the second part, the formulation commonly used to solve electromagnetic problem with the covariant method and the new contravariant method are introduced. In Section 3, these two methods are applied to the propagation analysis in a trapezoidal waveguide.

1. TENSORIAL FORMALISM

1.1. Definitions

A natural coordinate system, either orthogonal or not, to a surface S is such that one of the coordinates surfaces coincide with this surface S. In such system, the referential origin is defined by the intersection of the coordinates curves x_i = constant (for i=1…N). Let us consider a surface S separating two media and let $\vec{a}$ be a vector. If one calls X1 the coordinates curve x_1=constant and X2 for x_2=constant, then the covariant components a_i (i=1, 2) are tangent on the coordinates surfaces X_j (j=2,1) and the contravariant components are perpendicular to coordinates surfaces X_i (i=1,2) (Fig.1).

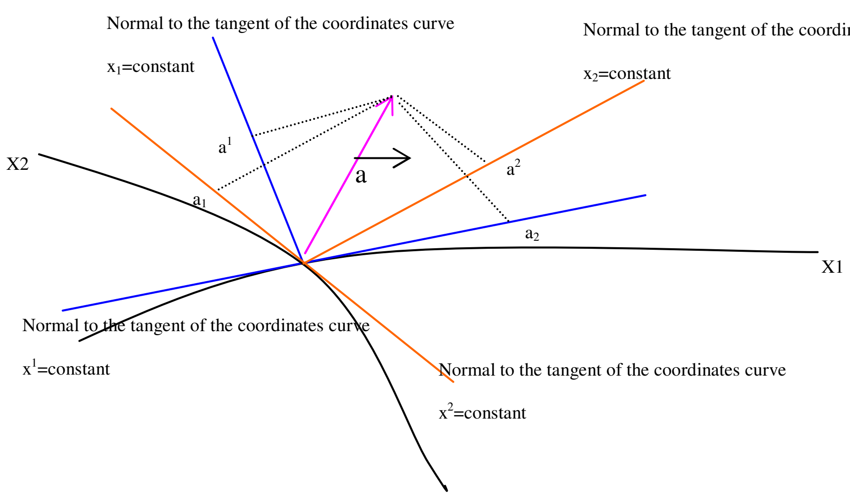


Fig. 1: Vector representation by these covariant and contravariant components.

1.2. Tensor Form For Maxwell Equation

In the present work, Maxwell's equations are used in a non orthogonal curvilinear coordinates. To achieve this, it is necessary to use tensorial calculus [2]. Among the various electromagnetism equations formulation available, the Maxwell-Minkowski-Post formalism has been chosen since it is invariant to a change of referential [3]. Their expressions and those of the constitutive relations are (1):

$$\xi^{ijk}\partial_j E_k = -\frac{\partial B^i}{\partial t} \qquad \partial_i D^i = \rho \qquad \text{and} \qquad D^i = \varepsilon^{ij} E_j \tag{1}$$

$$\xi^{ijk}\partial_j H_k = j^i + \frac{\partial D^i}{\partial t} \qquad \partial_i B^i = 0 \qquad\qquad B^i = \mu^{ij} H_j$$

where i, j, k $\in \{1\ 2\ 3\}$, E_k and H_k are, respectively, the covariant component of the electric field and the covariant component of the magnetic field. B^i and D^i are respectively the contravariant component of the magnetic induction and the electric flux density. ξ^{ijk} is the Levi-Civita indicator, whose only nonzero elements are

$$\xi^{123} = \xi^{231} = \xi^{312} = 1 \qquad \xi^{132} = \xi^{213} = \xi^{321} = -1$$

For a homogeneous, isotropic and linear medium of permittivity ε and permeability μ, the medium pseudo-tensors ε^{ij} et μ^{ij} which are twice contravariants are written:

$$\varepsilon^{ij} = \varepsilon\sqrt{g}\, g^{ij} \quad \mu^{ij} = \mu\sqrt{g}\, g^{ij} \quad \sqrt{g} = \det(g^{ij})$$

where $\mu=\mu_0$ and $\varepsilon=\varepsilon_0$ (the vacuum permeability and permittivity, respectively) and g^{ij} are the metric tensor components. Pseudo-tensors contain geometrical and physical information of the problem.

The boundary condition across the interface S (Fig.1) are written as $a_1(1) = a_1(2)$ (the tangential component are continuous) and $a^1(1) = a^1(2)$ (the normal component are continuous). In fact, the boundary conditions can be written on the interface between two media, such as on a plane, as soon as this interface coincides with a coordinates surface. For instance, the continuity of the tangential component of the electric field and the magnetic field on the surface X2 is expressed by the continuity of the covariant component E_i and the contravariant component B^i with $i \neq 2$.

2. COVARIANT AND CONTRAVARIANT METHOD

From the Maxwell-Minkowski equations and constitutive relations (1), and considering the load densities and current equal to zero in the medium, two formulations can be used, the covariant and the contravariant methods. The time dependence is assumed by the factor exp(iωt), where ω is the angular frequency, c and k are, respectively, the celerity and the wave number.

2.1. Covariant Method

Since the medium is considered linear, homogeneous and isotropic, many authors choose only one of the two electric and magnetic quantities, namely the covariant component of E and H, to study the electromagnetic behaviour of a waveguide or a grating [4]. The propagation equations of the electric field, for example, result from the systems (2) in Table 1. They are then written, for an index i=1 (the 2 other equations are obtained by circular shift) as given by (3) in Table 1.

2.2. Contravariant Method

Table 1: Analytical formulation of the covariant and contravariant methods.

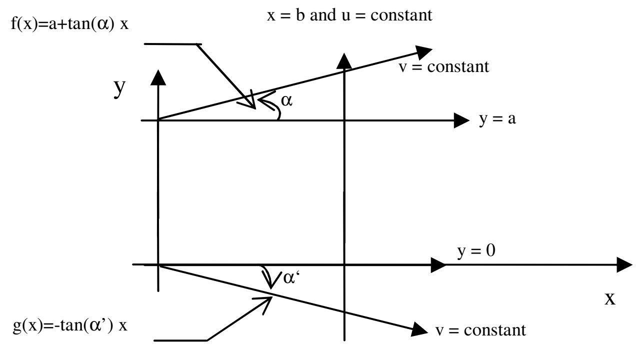
Covariant method	**Contravariant method**
$\xi^{ijk}\partial_j E_k = -ik\sqrt{g}\, g^{ij} ZH_j$ $\xi^{ijk}\partial_j ZH_k = ik\sqrt{g}\, g^{ij} E_j$ (2) with $Z = \sqrt{\mu/\varepsilon}$ Re(Z) > 0 and $k^2 = \omega^2\varepsilon\mu$ Re(k) > 0 $\partial_1\left[\frac{1}{\sqrt{g}}\left(g_{22}\partial_1 E_3 - g_{21}\partial_2 E_3\right)\right] + \partial_2\left[\frac{1}{\sqrt{g}}\left(g_{11}\partial_2 E_3 - g_{12}\partial_1 E_3\right)\right] + (k^2-\gamma^2)\sqrt{g}E_3 = 0$ $\partial_1\left[\frac{1}{\sqrt{g}}\left(g_{22}\partial_1 ZH_3 - g_{21}\partial_2 ZH_3\right)\right] +$ (3) $\partial_2\left[\frac{1}{\sqrt{g}}\left(g_{11}\partial_2 ZH_3 - g_{12}\partial_1 ZH_3\right)\right] + (k^2-\gamma^2)\sqrt{g}ZH_3 = 0$	$\xi^{ijk}\partial_j E_k = -i\omega B^i$ $\xi^{ijk}\partial_j g_{kl} B^l = i\frac{k}{c} g^{ij} E_j$ (4) $\Delta_t E_3 + k_c^2 E_3 = 0$ $\Delta_t \frac{B^3}{\sqrt{g}} + k_c^2 \frac{B^3}{\sqrt{g}} = 0$ (5) $k_c^2 = k^2 - \gamma^2$ with associated differential operator corresponding to the transverse Laplacian in curvilinear coordinates $\Delta_t = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}\, g^{ik}\partial_k\right).$

In a free-source medium, Maxwell's equations may be written as (4). Hence, the problem is restricted to a geometrically invariance along the z direction (i=3).
Longitudinal electromagnetic wave propagation is characterized by a propagation constant γ with $\partial_3 = -i\gamma$. The propagation's equations can be expressed as in (5).

3. APPLICATION

3.1. Geometry

These two methods have been applied to the propagation analysis in a waveguide with a constant trapezoidal section along the plan (xOy) (fig.2). To write the boundary conditions in a simple way, coordinates system such that the boundary surface coincides with a coordinates surface (u, v, w) is defined from the Cartesian system (x, y, z) (6).



$$\begin{cases} x \\ y \\ z \end{cases} \Rightarrow \begin{cases} u \\ v = \dfrac{a\left(y - g(x)\right)}{h(x)} \\ w \end{cases} \qquad (6)$$

Fig.2: Trapezoidal waveguide.

where $f(x) = a + \tan(\alpha)x$ is a function describing the higher profile, $g(x) = -\tan(\alpha')x$ the lower profile of the waveguide and h(x) = f(x) - g(x) .
The metric tensor g (7) is determined with a referential change which is associated to the geometry problem. Let M denotes a variable point referenced by the rectangular coordinates (x^i with i = 1, 2 or 3). At M, the so called natural referential (M, $e^{i'}$ with i' = 1, 2 or 3) is defined by the following basis vector (8):

$$g_{ij} = \begin{bmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad g_{i'j'} = A^i_{i'} A^j_{j'} g_{ij} \qquad g = \left(\frac{h}{a}\right)^2 \qquad (7)$$

$$\overrightarrow{e_{i'}} = A^i_{i'} \overrightarrow{e_i} \qquad A^i_{i'} = \frac{\partial x^i}{\partial x^{i'}} \qquad \overrightarrow{e_i} = A^{i'}_i \overrightarrow{e_{i'}} \qquad A^{i'}_i = \frac{\partial x^{i'}}{\partial x^i} \qquad (8)$$

The longitudinal components of the electromagnetic field are expressed in terms of the transversal ones from (2) and (4). The covariant and the contravariant field components are developed on a Fourier expansion basis. The propagation equations resolution (3) and (5) is made using the moment's method by projecting equations onto a Fourier basis. The solutions of this eigenvalues problem are the propagation constants and the longitudinal fields' amplitudes.

3.2. Covariant Method

One of the validation criteria retained, except the usual physical and energy criteria, is to compare our results with those of rectangular and triangular waveguides. The rectangular waveguide section dimensions are a/λ=0.47 and b/λ=0.97. In this waveguide, the only propagation mode is the fundamental one. The propagation constant value of the fundamental mode (theoretical value) is compared with the covariant theory.
The results obtained (fig.3) show us that the proposed formulation is not adequate for this particular problem. In fact, the formulation with the covariant component of E and H was largely exploited these last years [5]. But this one involves sometimes incoherencies in the results obtained. For a waveguide (the reasoning could be then applied to other propagating or diffracting structures), the metric tensor is not a scalar such as it has been conceived for a homogeneous, isotropic and linear medium in a Cartesian referential. Indeed, in this study, the permittivity and the permeability are expressed in a non-diagonal matrix form. The approximation usually made to consider only the field H with the boundary conditions of the field B is not valid any more. In a non-orthogonal curvilinear referential, a study meeting all the generally accepted standards requires that the couple (E, B) is considered. Their boundary conditions are simple to express. On the contrary, the boundary conditions for D and H cannot be easily written and considering that, the system is not orthogonal and the relations between E and D, and B and H are not linearly proportional.

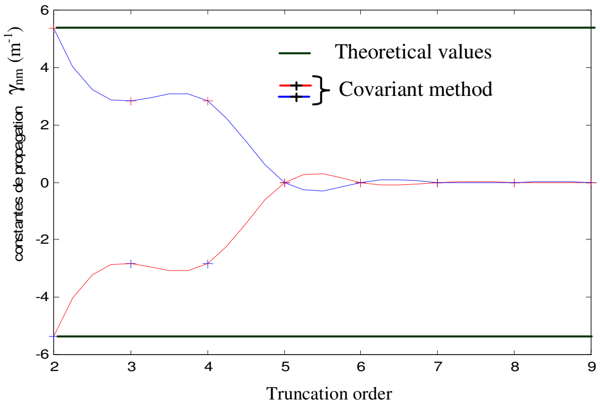


Fig.3: Variation of the imaginary part of the propagation constant
for the fundamental mode according to the truncation order for λ=1m.

3.3. Contravariant Method

To validate this method, a trapezoidal waveguide (fig.2) with dimensions a/λ=0.47, b/λ=0.97 for α = α'=2° is considered. The convergence of the pure imaginary constants values for a small truncation order (N=6) can be noted (fig.4). Another study is made for the transformation of a square guide (dimensions a/λ=0.97 and b/λ=0.97) to an isosceles right-angled triangular guide of size similar to the square one (fig.5). The only propagating mode found in this final structure is the mode TE01. The resulting code was validated considering results convergence according to the truncation order and compared with those yields by a commercial software with due regards to physical and numerical criteria.

4. CONCLUSION

Two methods using tensorial formalism have been compared to study propagation in non canonical waveguide. The usual method using covariant component is not adequate since the metric tensor is not a scalar in a non orthogonal curvilinear coordinates system and the boundary conditions of H are not easily expressed. The use of (E,B) is the only way to simulate correctly the propagation in a waveguide or the diffraction on a grating. The contravariant method can be used to analyse any structure with arbitrary shape.

Reference

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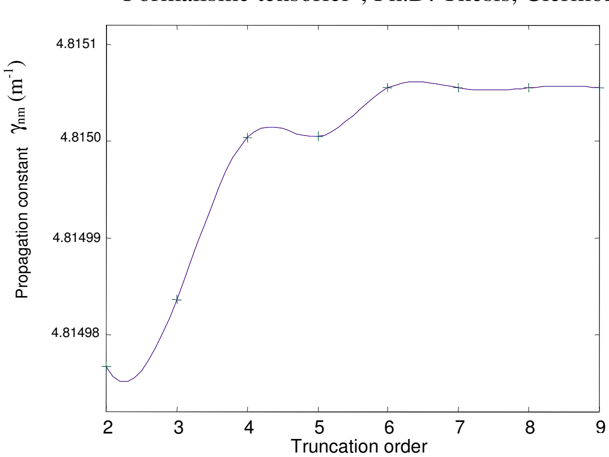

Fig.4: Propagation constant convergence according to truncation order.

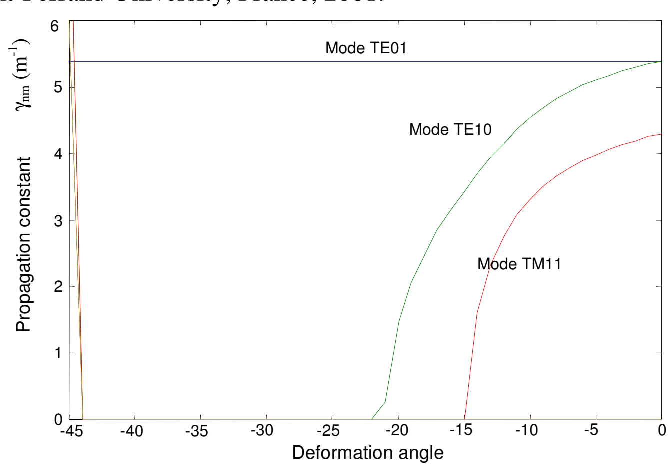

Fig.5: Variation of the propagation constant according to the deformation angle from a square to a triangle.