ABSTRACT

This work will focus on the extensions of the recently developed Multiplicative Regularized Contrast Source Inversion (MR-CSI) method to the full vectorial three-dimensional case, which up to now is not feasible by most available nonlinear inversion methods in the literature. As a test case we consider inversion of synthetic and experimental data obtained from dielectric objects at microwave frequency.

INTRODUCTION

Image reconstruction is a complicated nonlinear problem in microwave tomography because next to the material parameters, the field distribution in the investigation domain are also unknowns. Serious efforts have been made to solve this problem in two-dimensional model where the scalar Helmholtz equation (TM electromagnetic polarization) can be applied. The Newton method has been successfully applied to high contrast objects for two-dimensional model only. The main bottleneck of this Newton approach, especially discouraging in the three-dimensional case, is the need of multiple forward solutions, needed to construct the Hessian matrix. The same problem has also occurred in the distorted Born method or its equivalent Newton-Kantorovich. This computational problem makes that up to today only a few reports with respect to image three-dimensional objects have been published. But most of them only consider either a weak scattering object using vector Maxwell equations or a strong scattering object employing a scalar Helmholtz equation.

Another type of method which avoids solving any forward problem in each iterative step is the so-called Modified Gradient method where the contrast and the fields are updated simultaneously in each iterative step using Conjugate Gradient directions. Later on it was shown that by updating the contrast sources (the product of the contrast with the fields) and the contrast, one can arrive at a simpler algorithm than the Modified Gradient method while maintaining its robustness and abilities. This method is called the Contrast Source Inversion (CSI) method. Recently, this CSI method has been armed with a total variation regularizer in order to handle high contrast objects. Although the addition of the total variation to the cost functional has a very positive effect on the quality of the reconstructions for both ‘blocky’ and smooth profiles, a drawback is the presence of an artificial weighting parameter in the cost functional, which can only be determined through considerable numerical experimentation and a priori information of the desired reconstruction. Therefore, it was suggested to include the total variation as a multiplicative constraint, with the result that the original cost functional is the weighting parameter of the regularizer, so that this parameter is determined by the inversion procedure itself. The multiplicative type of regularization seems to handle noisy as well as limited data in a robust way without the usually necessary a priori information. In this paper the CSI method using the new weighted $L^2$-norm total variation regularizer is applied to handle more complicated objects in full three-dimensional medium.

INTEGRAL REPRESENTATIONS

We consider an unknown object, $B$, of arbitrary bounded cross section, not necessary connected. Let $D$ denote the interior of a bounded domain with piecewise smooth discontinuity interfaces. A Cartesian coordinate system is centered in $D$ with spatial points denoted by $x$. We assume that the unknown scatterer, $B$, is contained in the object domain $D$. The fields are assumed to vary sinusoidally in time with frequency $\omega$, time factor $\exp(-i\omega t)$ where $i^2 = -1$ and $t$ is the time parameter. The corresponding wavelength is denoted by $\lambda$. If the vector $E_j^{inc}$ denotes an incident electric field with source located at $x_j^S$, then for each incident field, the total electric field $E_j$ in $D$ is given by $E_j(x) = E_j^{inc}(x) + E_j^{sct}(x)$. It is well known that the total field $E_j$ in $D$ satisfies the
following domain integral equation:

\[ E_j^{\text{inc}}(x) = E_j(x) - [k_b^2 + \nabla \nabla] \int_D G(x - x') \chi(x') E_j(x') \, dv(x'), \quad x \in D, \]

where \( G(x) = \exp(i k_b |x|)/(4\pi |x|) \) with \( k_b = \omega/\sqrt{\varepsilon_0 \mu_0} \) and \( \varepsilon_0 \mu_0 > 0 \) in which \( \mu_0 \) is the vacuum permeability and \( \varepsilon_0 \) is the complex permittivity of the background medium. In (1), the symbol \( \nabla \) denotes the spatial differentiation with respect to the position vector \( x \). Further the function \( \chi \) denotes the contrast of the material properties of the unknown object with respect to the background medium, given by \( \chi(x) = \varepsilon(x)/\varepsilon_0 - 1 \). In the inverse scattering problem the scattered electric field vector \( E_j^{\text{scat}} \) will be measured on the data domain \( D \) outside \( D \). The data domain \( S \) can be a discrete collection of points, surfaces or volumes. The integral representation for the scattered electric field vector \( E_j^{\text{scat}} \) is given by

\[ E_j^{\text{scat}}(x) = [k_b^2 + \nabla \nabla] \int_D G(x - x') \chi(x') E_j(x') \, dv(x'), \quad x \in S. \]

In order to discuss our solution of the inverse scattering problem we write our equations in an operator form. Equation (2) is called the data equation and (1) is called the object equation, they are written symbolically as

\[ E_j^{\text{scat}} = \mathcal{K}_S \chi E_j, \quad x \in S \quad \text{and} \quad E_j^{\text{inc}} = E_j - \mathcal{K}_D \chi E_j, \quad x \in D. \]

The inverse scattering problem can now be formulated as follows: finding \( \chi \) of the object domain \( D \) for given \( E_j^{\text{scat}} \) at the data domain \( S \), or solving the data equation for \( \chi \), subject to the additional and necessary condition that \( \chi \) and \( E_j \) satisfy the object equation.

**INVERSION ALGORITHM**

A major observation is that the data and object equations contain both the unknown field and the unknown contrast in the form of a product; it can be written as a single quantity, viz. the contrast source, \( W_j(x) = \chi(x) E_j(x) \). In the present inversion method, one chooses to reconstruct the contrast sources and the contrast instead of the fields and the contrast. Using the contrast source definition, the data and object equations become

\[ E_j^{\text{scat}} = \mathcal{K}_S W_j, \quad x \in S \quad \text{and} \quad \chi E_j^{\text{inc}} = W_j - \chi \mathcal{K}_D W_j, \quad x \in D. \]

The so-called Multiplicative Regularized CSI (MR-CSI) method, consists of an algorithm to construct sequences of \( \{W_j,n\} \) and \( \{\chi,n\} \) \((n = 1, 2, \cdots)\) which iteratively reduce the value of the cost functional,

\[ F_n(W_j, \chi) = \left[ F^S(W_j) + F^D_n(W_j, \chi) \right] F^R_n(\chi), \]

where

\[ F^S(W_j) = \eta^S \sum_j \| E_j^{\text{scat}} - \mathcal{K}_S W_j \|^2_S, \quad \eta^S = \sum_j \| E_j^{\text{scat}} \|_S^2 \]

\[ F^D_n(W_j, \chi) = \eta^D_n \sum_j \| \chi E_j^{\text{inc}} - W_j + \chi \mathcal{K}_D W_j \|_D^2, \quad \eta^D_n = \sum_j \| \chi_n - 1 E_j^{\text{inc}} \|_D^2, \]

\[ F^R_n(\chi) = \int_D b_n^2 (|\nabla \chi|^2 + \delta_n^2) \, dv(x) \quad b_n = [\int_D dv(x)]^{-1/2}(|\nabla \chi_{n-1}|^2 + \delta_n^2)^{-1/2}. \]

Although the constant parameter \( \delta_n^2 \) is introduced for restoring differentiability of the regularizer, it also controls the influence of the regularization. We therefore have chosen to increase the regularization as a function of the number of iterations by decreasing this parameter \( \delta_n^2 \). Since the normalized object error term will decrease as a function of the number of iterations, we choose

\[ \delta_n^2 = F^D_n(W_j,n, \chi_n-1)(\Delta x)^2, \]

where \( \Delta x \) denotes the side-length of the mesh size of the discretized domain \( D \). Its choice is inspired by the idea that in the first few iterations, we do not need the minimization of the regularizer and as the iterations...
proceed we want to increase the effect of the regularizator. The structure of the cost functional in (5) is such that it will minimize the factor $F_R^R$ with a large weighting parameter in the beginning of the optimization process, because the value of $F_S + F_D^R$ is still large, and that it will gradually minimize more and more the normalized errors in the data and object equations when the factor $F_R^R$ remains a nearly constant value close to one. If noise is present in the data, the term $F_S$ will remain at a large value during the whole optimization and therefore, the weight of the factor $F_R^R$ will be more significant. Hence, the noise will, at all times, be suppressed in the reconstruction process.

The algorithm starts with back propagation as the initial estimates for the contrast sources and the contrast. Then, in each iteration, we first update the contrast sources $W_{j,n}$ using one conjugate gradient step

$$W_j = W_{j,n-1} + \alpha_n^w w_{j,n}, \tag{10}$$

where the functions $w_{j,n}$ is the Polak-Ribière conjugate gradient directions and the real parameter $\alpha_n^w$ is found as minimizer of $F^R_{R}(W_{j,n-1} + \alpha_n^w w_{j,n}, \chi_{n-1})$. Note that the cost functional in (5) is a quadratic function of $\alpha_n^w$, and only one minimizer is arrived at. In the updating scheme for the contrast sources, the contrast quantity $\chi$ set to be equal to $\chi_{n-1}$, then $F^R_R(\chi_{n-1})$ is always equal to one during the updating of $W_{j,n}$. After the contrast sources have been obtained, we compute the field $E_{j,n}$ by substituting (10) in $E_{j,n} = E^{inc} + K_D W_{j,n}$.

In the second part of each iteration, we update the contrast for given contrast sources. Unlike in the case of updating the contrast sources, the cost functional in (5) is a nonlinear function in term of the contrast. Then, for updating the contrast, we proceed in two steps: First, we observed that the closed-form expression of the contrast can be found if the regularization factor $F^R_R$ is absent, viz.,

$$\chi_n = \arg\min_{\text{complex}} \chi \left\{ F^S(W_{j,n}) + F^D_R(W_{j,n}, \chi) \right\} = \sum_j W_{j,n} \cdot \overline{E_{j,n}}/\sum_j |E_{j,n}|^2. \tag{11}$$

From this point we make an additional minimization step to take into account the effect of the regularization factor,

$$\chi_n^R = \chi_n + \alpha_n^R d_n \quad \text{with} \quad d_n = g^R_n + \text{Re}(g^R_n / g^R_{n-1}) D d_{n-1} \tag{12}$$

where $\chi_n$ is now given by (11) and $d_n$ is the Polak-Ribière conjugate gradient. We remark that we prefer now a line minimization around the minimum of the cost functional $F^S + F^D_R$, which represent the problem. In view of (11) we take $g^R_n$ as

$$g^R_n = \left[ F^S(W_{s,n}) + F^D_R(W_{s,n}, \chi_n) \right] \nabla \cdot (b_n^2 \nabla \chi_n) / \sum_j |E_{j,n}|^2 \tag{13}$$

being a preconditioned gradient of the cost functional $F_n(W_{s,n}, \chi)$ with respect to changes in the contrast around the point $\chi = \chi_n$. In view of the previous minimization step, the gradient of $F^D_R$ with respect to changes in the contrast around the point $\chi = \chi_n$ vanishes. Hence, the gradient with respect to the contrast, in contrary to the previous approaches of the MR-CSI method, contains only a contribution of the regularization additionally imposed. The real parameter $\alpha_n^R$ is found from a line minimization as minimizer of cost functional in (5). The minimization of $F_n(W_{j,n}, \chi_n + \alpha_n^R d_n)$ can be performed analytically which is a fourth-degree polynomial in $\alpha$, $F_n(\alpha) = [A + B(\alpha)^2] [X + 2Y \alpha + Z(\alpha)^2], \tag{14}$

$$X = \|b_n \nabla \chi_n\|^2_D + \delta_n^2 \|b_n\|^2_D, \quad A = F^S(W_{j,n}) + F^D_R(W_{j,n}, \chi_n),$$

$$Y = \text{Re}(b_n \nabla \chi_n \cdot b_n \nabla d_n) / D, \quad B = \sum_j |d_n E_{j,n}|^2 / \sum_j |\chi_{n-1} E_{j,n}^{inc}|^2_D,$$

$$Z = \|b_n \nabla d_n\|^2_D.$$
Further the term $AZ^2/B$ is always non-negative, hence, a sufficient condition that the second derivative is a positive function of $\beta$ is $Y^2 \leq \frac{2}{X} XZ$ or $|\text{Re}(b_n \nabla \chi, b_n \nabla d_n)_D|^2 / (||b_n \nabla \chi||_D^2 + \delta_n^2 ||b_n \nabla d_n||_D^2) \leq 2/3$. However, from Cauchy-Schwarz inequality we know that $\text{Re}(b_n \nabla \chi, b_n \nabla d_n)_D^2 / (||b_n \nabla \chi||_D^2 ||b_n \nabla d_n||_D^2) \leq 1$. This means that

$$\delta_n^2 \geq ||b_n \nabla \chi_n||_D^2 / (2 ||b_n ||_D^2)$$

is a sufficient condition for the cost functional to be a convex function with one minimum. If the choice for the parameter $\delta_n^2$ of (9) is less than the right-hand side of (15), we replace the value of $\delta_n^2$ by the right-hand side of (15). We have refrained from using this value for $\delta_n^2$ in the whole iteration procedure, because from our numerical observations it appears this value has a large variation in the beginning of optimization procedure.

After we have obtained a new estimate $\chi_n^R$ for the contrast, we repeat again the updating of contrast sources $W_{j,n}$ (if the value of the cost functional is not small enough) starting with $\chi_{n-1} = \chi_{n-1}^R$ of the previous iteration.

**NUMERICAL RESULTS**

As an example, we consider inversion of real dataset which is measured by the Institute Fresnel in Marseille, France. The experimental setup of the Institut Fresnel used for measuring the data consists of a transmitting antenna which generates an electromagnetic wave field at various measurement frequencies and a receiving antenna which is of the same type. The objects are all very large in the direction perpendicular to the plane in which the antennas are located. Therefore a two-dimensional TM electromagnetic inversion model is allowed, but in this work we will use this data set at a particular frequency to test our full vectorial three-dimensional nonlinear inversion scheme. The center of the plane of illumination is the center of our test domain $D$, therefore is the center of our Cartesian system. The transmitting antenna illuminates the objects from radius of 0.72 m at 36 different locations distributed equidistantly around the object in the plane $x_3 = 0$. The receiving antenna measures the total and the incident field at 72 different locations distributed equidistantly over a circle with a radius of 0.76 m around the object in the plane $x_3 = 0$. Due to physical limitations there is a minimal angle between the transmitting and receiving antenna such that for each illumination angle the field is measured for 49 out of the 72 receiver angles.

The dataset (twodieITM-8f.exp) is generated for $s$ scatterer containing of two dielectric cylinders with circular cross-section of radius 15 mm. These cylinders are placed at about 30 mm with respect to the center of the experimental setup. The permittivity of this twin dielectric cylinder is supposed to be $\varepsilon/\varepsilon_1 = 3 \pm 0.3$ ($\chi = 2 \pm 0.3$). Eight frequency are measured from one up to eight GHz. We use only the data at 4 GHz frequency. In the inversion the cubic test domain $D$ is discretized into $45 \times 45 \times 15$ subcubes with side length of 3.75 mm. The inversion results at the plane $x_3 = 0$ after 128 iterations using the scalar (where we have neglected the grad-div operators in the data and object equations) and vectorial formulation are given in Fig. 1. Obviously by using the full vectorial three-dimensional formulation we are able to obtain the shape, location and the value of the material parameter of the unknown object using a single frequency data.

Figure 1: Reconstructed real part of the contrast of the twin dielectric cylinder in the plane $x_3 = 0$ from an experimental dataset using a scalar 2D (a) and a vectorial 3D (b) formulation.