Analytical study of Virtual Experiments via Linear Sampling Method for 2D dielectric targets

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Abstract

In this contribution the Virtual Experiments paradigm as applied to the solution of the canonical inverse scattering problem in the 2D scalar case is addressed to show present performance and limitations, and possibly give some insights to overcome these latter. It will be shown, through analytic findings and numerical results (shown at the conference), the usefulness of such a study in the framework of inverse scattering problems.

1 Introduction

The electromagnetic inverse scattering problem (ISP) represents an open issue from both methodological and application point of view since it requires solution of a nonlinear and ill-posed problem through reliable and low computational cost techniques to be exploited in several applications [1]. For the above considerations, lots of solution schemes have been proposed in the relevant literature, whose review is out of scope for a conference paper.

In some papers over the last years, the authors contributed to introduce a new approach for solving the ISP: the Virtual Experiments (VEs). This framework amounts to recast the original scattering experiments into new virtual (software defined) ones, through proper design equations mainly related to physical interpretation of sampling approaches introduced for the inverse obstacle problem, such as the Linear Sampling Method [2]. Indeed, when VEs are built via LSM as design equation [3], circular symmetry of the induced contrast sources, and scattered field, can be enforced for a set of sampling points (namely pivot points) which belong to the scatterer's support. As result, such a kind of internal field conditioning can be conveniently exploited in the inversion stage in many different effective ways as demonstrated in [4, 5, 6]. The VEs has shown a great capability in imaging targets which do not belong to the well known weak scattering regime (Born approximation). However, its performance get worsen and worsen when target's dimensions and dielectric contrast increase.

To understand the range of applicability and the possibility to improve the method's performance, in this paper, an analytical formulation of the problem is introduced and discussed for the case of a dielectric circular cylinder. The contribution is divided in the following sections. In Section 2, the main mathematical equations are introduced, while in Section 3 and 4 the main outcomes of this study are discussed. Numerical assessment towards benchmark examples will be given at the conference.

2 Formulation of the problems

Let us introduce the scattering equations for any actual virtual experiment as follows:

$$\mathscr{E}_s^p(\underline{r}_m) = k^2 \int_\Omega g(\underline{r}_m, \underline{r}')\chi(\underline{r})\mathscr{E}_t^p(\underline{r}')d\underline{r}', \qquad \underline{r}_m \in \Gamma, \quad p = 1, \ldots, P \tag{1}$$

$$\mathscr{E}_t^p(\underline{r}) - \mathscr{E}_i^p(\underline{r}) = k^2 \int_\Omega g(\underline{r}, \underline{r}')\chi(\underline{r}')\mathscr{E}_t^p(\underline{r}')d\underline{r}', \qquad \underline{r} \in \Omega, \quad p = 1, \ldots, P \tag{2}$$

wherein $\mathscr{E}_s^p$, $\mathscr{E}_t^p$, $\mathscr{E}_i^p$ are the virtual scattered, total and incident fields, respectively; χ is the contrast function, encoding the target properties; k is the wave number; $\underline{r}$ scans the investigation domain Ω; $\underline{r}_m$ denotes the measurements points; g is the Green's function pertaining to the background medium. The parameter superscript p stands for some given sensing diversity, as commonly done for the illumination diversity in the multi-view measurement scheme. For the case at hand it stands for different pivot points. Accordingly, the rearranged virtual quantities reads as:

$$\begin{aligned}
\mathscr{E}_s^p(\underline{r}_m) &= \sum_{v=1}^{V} c(\theta_v)E_s(\theta_v, \underline{r}_m) \\
\mathscr{E}_t^p(\underline{r}) &= \sum_{v=1}^{V} c(\theta_v)E_t(\theta_v, \underline{r}) \\
\mathscr{E}_i^p(\underline{r}) &= \sum_{v=1}^{V} c(\theta_v)E_i(\theta_v, \underline{r})
\end{aligned} \tag{3}$$

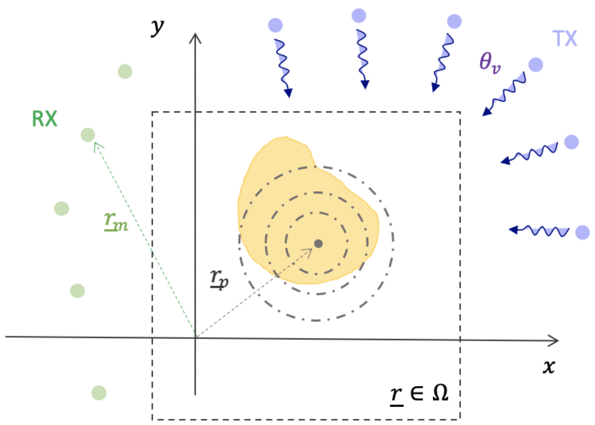

Figure 1. Sketch of the problem at hand.

wherein E_s, E_i and E_t are the scattered, incident and total fields in the orginal multi-view experiments; $c(\theta_v)$ are given superposition excitation coefficients for the v-th antenna (with view angle θ_v, see Figure 1 for a sketch of the problem at hand) to be determined by means of a smart pre-processing. The most prominent advantage of considering equations (1)-(2) in place of the original ones (formally the same) is that prior information about the internal field $\mathscr{E}_t^p(\underline{r})$ can be gained by means of such pre-processing strategy.

To achieve a rearrangement useful for the inversion process, a proper design equation acting on measured scattered fields has to be adopted to determine $c(\theta_v)$. An effective possibility is offered by the far field equation [3], that is the equation underlying the linear sampling method. This equation amounts to solve the following linear problem:

$$\mathscr{F}[c(\theta_v,\underline{r}_s)] = \int_\Gamma E_s(\underline{r}_m,\theta_v)c(\theta_v,\underline{r}_s)d\theta_v = g(\underline{r}_m,\underline{r}_s) \quad (4)$$

wherein $\mathscr{F}[\cdot] : L^2(\Gamma) \to L^2(\Gamma)$ is the so called far field operator, $c(\theta_v,\cdot)$ and $g(\cdot,\underline{r}_m)$ are, respectively, the problem unknown and the field radiated by an elementary source for each r_s points belonging to an arbitrary grid that samples the region of imaging Ω.

Since $\mathscr{F}$ is a compact operator, solution of (4) can be pursued only through a regularized approach, then by simply plotting the solution norm $||c||_{L^2}$ (usually in a logarithmic scale) over the sampling grid allows to estimate the support of a large class of targets, since $||c||_{L^2}$ attains its lowest values inside the target's support [3].

On the other hand, if we consider a proper subset of sampling points inside the support of the scatterer (pivot points r_p) and recast the original scattering equations (1)-(2) by means of (3), the simultaneous (virtual) excitation of the primary fields enforces the scattering system to behave like a point-like scatterer. This is actually feasible when the internal field and contrast source show a given specific distribution in the scatter's support. In the next section we study the internal virtual field expression for the case of a dielectric cylinder through analytical formulas.

3 Virtual internal field for a dielectric cylinder

According to [7], the transmitted (internal) field E_t for a penetrable dielectric cylinder can be analytically expanded in terms of cylindrical harmonics with given expansion coefficients taking into account dimension and dielectric properties of the cylinder. On the other hand, also an analytical solution of the FFE can be pursued in terms of Fourier series expansion in case of circular dielectric cylinder [8].

As a result, it is possible to rewrite the internal field through the analytic solution of the FFE and the cylinder's modal expansion. After simple algebra, the final form of such a field for any given pivot point reads as:

$$\mathscr{E}_t^p(\underline{r},\underline{r}_p) = \sum_{m=-\infty}^{+\infty} b_m J_m(k_s r) J_m(k_0 r_p) e^{jm(\theta_r-\theta_p)} \quad (5)$$

wherein $\underline{r} = r(cos\theta_r, sin\theta_r)$, $\underline{r}_p = r_p(cos\theta_p, sin\theta_p)$ and b_m is given by:

$$b_m = \frac{J_m(k_0a)H_m'^{(2)}(k_0a) - J_m'(k_0a)H_m^{(2)}(k_0a)}{\sqrt{\varepsilon_s}J_m(k_0a)J_m'(k_sa) - J_m'(k_0a)J_m(k_sa)}. \quad (6)$$

ε_s and a being dielectric constant and radius of the cylinder, respectively, k_s and k_0 the propagation constants in the medium and in the vacuum, respectively, and $J(\cdot)$, $H(\cdot)^{(2)}$, $J(\cdot)'$, $H(\cdot)'^{(2)}$ the Bessel and 2-th kind Hankel functions and their first derivatives, respectively.

The above formula give some useful hints to understand the spatial behavior of the internal field. First of all it is important to notice that when $\underline{r}_p = (0,0)$, the only term in (5) is the one for $m = 0$ as $J_m(0) = 0$ for $m \neq 0$. Then, the VE total field induced in the cylinder is exactly a Bessel function of zero order apart from the unessential constant b_0. On the other hand, when $\underline{r}_p \neq (0,0)$ the internal field strongly depends on the values of the expansion coefficients. In order to gain more insights in such an expression, let us firstly suppose that the coefficients b_m are constant with respect to the order m (at least for the relevant terms equal to $M \approx k_s a$). Under this assumption, the expression (5) can be further simplified by exploiting the Graf's addition theorem for the Bessel functions [9] to get:

$$\mathscr{E}_t(\underline{r},\underline{r}_p) \approx b_0 J_0\left(k_s\left|\underline{r} - \frac{\underline{r}_p}{\sqrt{1+\chi}}\right|\right). \quad (7)$$

The above expression states that the virtual total field can be approximated (apart from an unessential constant factor) by a 0-th order Bessel function. Hence the virtual total field exhibits a circular symmetry and is still focused, but not simply with respect to the pivot point $\underline{r}_p$ (as one would expect from the cylindrical pattern enforced by the right-hand-term of the FFE) since a shift of the focusing point is introduced due to the scale factor $(\sqrt{1+\chi})^{-1}$.

Finally, if the coefficients b_m exhibit a non constant behavior with respect to the order m, an exact zero order Bessel function cannot be achieved by means of equation (5). Plots of b_m for different values of cylinder's radius and dielectric constant reveal that the coefficients' value is almost constant with m up to a given value of $k_s a$, otherwise it does not.

4 Main outcomes of the analytic findings and implications for the imaging task

The above analysis reveals some interesting features related to the VEs settings as well as strategies to improve their performance, which can be summarized as follows:

- the study of the coefficients b_m states that when the dielectric properties of the target are much different from those of the background medium and/or its electric dimension comparable greater than the probing wavelength, the values of b_m are not constant with the index m. Accordingly, VEs enforcing a perfect circular symmetry of the internal field can be pursued only for a given range of optical density and electrical dimensions of the targets. This item underlines the effectiveness of the already available VE based approaches (both within linear and non linear solution strategies) in handling a class of target which is anyway well beyond the weak scattering regime approximation [2, 4, 5, 6];

- it is possible to enforce a perfect circular symmetry by the internal total field whatever dimension and optical density of the scatterer only for the center of the target. All the other pivot points introduce a shift of the focusing point that depends on the contrast itself; this item suggests possible "modal diversity" VE-based approaches recasting several experiments only for one pivot at the center of the scattering system.

- for very high contrasted and electrical large scattering systems, the behavior of the expansion coefficients b_m entails a total field which is neither focused nor circular symmetric with respect to the neighborhood of the sampling point; this item suggests that for high contrast targets a new design equation must be introduced taking into account field structure inside the scattering system almost different from the focused circular symmetric one.

The above considerations give some useful information to overcome limitations of the LSM based VE design equation for quantitative imaging of targets. Starting from the above outcomes, some effective strategies will be proposed to counteract the actual drawbacks and improve performance. Numerical results will be shown at the conference to corroborate the developed analysis with the aim to enlarge the class of retrievable targets via VEs.

References

[1] M. Pastorino, *Microwave Imaging*. Wiley Online Library, 2010.

[2] L. Crocco, I. Catapano, L. Di Donato, and T. Isernia, "The linear sampling method as a way for quantitative inverse scattering," *IEEE Trans. Antennas Propag.*, vol. 4, no. 60, pp. 1844–1853, 2012.

[3] D. Colton, H. Haddar, and M. Piana, "The linear sampling method in inverse electromagnetic scattering theory," *Inverse Probl.*, vol. 19, pp. 105–137, 2003.

[4] L. Di Donato, M. Bevacqua, L. Crocco, and T. Isernia, "Inverse scattering via virtual experiments and contrast source regularization," *IEEE Trans. Antennas Propag.*, vol. 63, no. 4, pp. 1669 – 1677, 2015.

[5] M. Bevacqua, L. Crocco, L. Di Donato, and T. Isernia, "An algebraic solution method for nonlinear inverse scattering," *IEEE Trans. Antennas Propag.*, vol. 63, no. 2, pp. 601 – 610, 2015.

[6] R. Palmeri, M. T. Bevacqua, L. Crocco, T. Isernia, and L. Di Donato, "Microwave imaging via distorted iterated virtual experiments," *IEEE Transactions on Antennas and Propagation*, vol. 65, no. 2, pp. 829–838, 2016.

[7] D. S. Jones, *Acoustic and Electromagnetic waves*. Oxford: Clarendon Press, 1989.

[8] A. Liseno and R. Pierri, "Impossibility of recovering a scatterer's shape by the first version of the linear sampling method," *Int. J. Electron. Commun.*, vol. 57, no. 1, pp. 70–73, 2003.

[9] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions With Formulas, Graphs and Mathematical Tables*. New York: Dover, 1972.