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Scalar Potential Formulation in Spherical Coordinates

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Abstract

A scalar potential formulation for an anisotropic gyrotropic medium containing sources is developed. The formulation results in two forced coupled scalar Helmholtz-like equations. The results simplify to well-known relations upon specialization to an isotropic medium. A radial magnetically-biased ferrite is proposed for possible application manifesting non-reciprocal behavior for controlling electromagnetic radiation and scattering.

1. Introduction

Advances in manufacturing makes the fabrication of ever exotic media and devices possible. Materials having gyrotropic and non-reciprocal properties, especially at optical frequencies, are of current interest [1]. Although gyrotropic materials are often used in planar environments, there is interest in other geometries, including spherical coordinates [2]. A scalar potential formulation is developed here to provide a simplifying framework for studying spherical gyrotropic media, including the presence of sources. This theoretical formulation is developed in Section 2. Salient remarks are made in Section 3. Section 4. summarizes the effort and concludes with future work, including a brief discussion of a possible application.

2. Scalar Potential Formulation

A scalar potential formulation for a spherically gyrotropic homogeneous medium containing sources is developed by first considering Faraday's law (and Ampere's law via duality) in the frequency domain, namely

$$\nabla\times\vec{E} = -\vec{J}_h - j\omega\vec{B}\,, \tag{1}$$

with $e^{j\omega t}$ time dependence assumed and suppressed. Note, the dependence on $(\vec{r},\omega)$ for the fields and current density is dropped for notational convenience. If the medium is a radially biased ferrite, for example, the constitutive relation becomes $\vec{B} = \vec{\mu}(\omega)\cdot\vec{H}$, where the permeability tensor is given by

$$\vec{\mu} = \begin{bmatrix} \mu_r & 0 & 0 \\ 0 & \mu_t & -j\mu_g \\ 0 & j\mu_g & \mu_t \end{bmatrix} = \hat{r}\hat{r}\mu_r + \vec{I}_t\mu_t + \hat{r}\times\vec{I}_t j\mu_g\,, \tag{2}$$

where $\vec{I}_t = \hat{\theta}\hat{\theta} + \hat{\phi}\hat{\phi}$, and μ_r , μ_t , and μ_g are the radial, transverse, and gyromagnetic permeability parameters, respectively. Thus, Faraday's law is written as

$$\nabla\times\vec{E} = -\vec{J}_h - j\omega(\hat{r}\hat{r}\mu_r + \vec{I}_t\mu_t + \hat{r}\times\vec{I}_t j\mu_g)\cdot\vec{H}\,. \tag{3}$$

The next step towards obtaining the scalar potential formulation is expanding the fields and current densities into radial and transverse parts with subsequent application of the 2D Helmholtz theorem [3], namely

$$\vec{E} = \hat{r}E_r + \vec{E}_t = \hat{r}E_r + \nabla_t\Phi + \nabla_t\times(\hat{r}\Theta)\,, \tag{4}$$

$$\vec{H} = \hat{r}H_r + \vec{H}_t = \hat{r}H_r + \nabla_t\Pi + \nabla_t\times(\hat{r}\Psi)\,, \tag{5}$$

$$\vec{J}_e = \hat{r}J_{er} + \vec{J}_{et} = \hat{r}J_{er} + \nabla_t U_e + \nabla_t\times(\hat{r}V_e)\,, \tag{6}$$

$$\vec{J}_h = \hat{r}J_{hr} + \vec{J}_{ht} = \hat{r}J_{hr} + \nabla_t U_h + \nabla_t\times(\hat{r}V_h)\,, \tag{7}$$

where Φ,Θ,Π , and Ψ are scalar potentials, and U_e, V_e, U_h , and V_h are scalar current density fields. Note, the transverse gradient terms are lamellar and the transverse curl terms are solenoidal. Substituting (4), (5), and (7) into (3), and using orthogonality of the radial and transverse components leads to

$$\begin{aligned}\nabla\times[\hat{r}E_r + \nabla_t\Phi + \nabla_t\times(\hat{r}\Theta)] &= \\ -\hat{r}J_{hr} - \nabla_t U_h - \nabla_t\times(\hat{r}V_h) & \\ -\hat{r}j\omega\mu_r H_r - j\omega\mu_t\nabla_t\Pi - j\omega\mu_t\nabla_t\times(\hat{r}\Psi) & \\ +\omega\mu_g\hat{r}\times\nabla_t\Pi + \omega\mu_g\hat{r}\times\nabla_t\times(\hat{r}\Psi). &\end{aligned} \tag{8}$$

By vector identity, the term $\nabla_t\times(\hat{r}\psi)$ is equivalent to $-\hat{r}\times\nabla_t\psi = -\hat{r}\times\nabla\psi = \nabla\times(\hat{r}\psi)$, where ψ is a generic scalar field. Based on these equivalent representations, Faraday's law in (8) becomes

$$\begin{aligned}\nabla\times[\hat{r}E_r + \nabla_t\Phi + \nabla\times(\hat{r}\Theta)] &= \\ -\hat{r}J_{hr} - \nabla_t U_h + \hat{r}\times\nabla_t V_h & \\ -\hat{r}j\omega\mu_r H_r - j\omega\mu_t\nabla_t\Pi + j\omega\mu_t\hat{r}\times\nabla_t\Psi & \\ +\omega\mu_g\hat{r}\times\nabla_t\Pi + \omega\mu_g\nabla_t\Psi. &\end{aligned} \tag{9}$$

It is important to note that the right hand side of (9) is represented by radial ($\hat{r}$) terms, transverse lamellar (∇_t) terms, and transverse solenoidal ($\hat{r}\times\nabla_t$) terms.

The physical nature of the left hand side of (9) is determined via the curl operation, leading to

$$\nabla\times(\hat{r}E_r) = -\hat{r}\times\nabla_t E_r\,, \tag{10}$$

$$\nabla\times\nabla_t\Phi = \hat{r}\times\nabla_t\left(\frac{\partial\Phi}{\partial r}\right), \tag{11}$$

$$\nabla \times [\nabla \times (\hat{r}\Theta)] = -\hat{r}\nabla_t^2 \Theta + \nabla_t \left(\frac{\partial \Theta}{\partial r} \right). \tag{12}$$

Thus, the terms in (10) and (11) are transverse rotational, and the terms in (12) are radial and transverse lamellar, respectively. Since the radial, transverse lamellar, and transverse rotational components are orthogonal (which may be deduced via Fourier transformation, for example, with the assumption of decaying potentials), their linear independence allows these respective components to be equated on the left and right-hand sides of (9), leading to the following relations

$$\nabla_t^2 \Theta - J_{hr} - j\omega\mu_r H_r = 0\,, \tag{13}$$

$$\nabla_t \left(\frac{\partial \Theta}{\partial r} + U_h + j\omega\mu_t \Pi - \omega\mu_g \Psi \right) = 0\,, \tag{14}$$

$$\hat{r} \times \nabla_t \left(E_r - \frac{\partial \Phi}{\partial r} + V_h + j\omega\mu_t \Psi + \omega\mu_g \Pi \right) = 0\,. \tag{15}$$

The relations in (14) and (15) show that the terms inside the parentheses could, in general, be scalar functions of the radial variable r, but constant with respect to θ and ϕ, that is,

$$\frac{\partial \Theta}{\partial r} + U_h + j\omega\mu_t \Pi - \omega\mu_g \Psi = C_1(r)\,, \tag{16}$$

$$E_r - \frac{\partial \Phi}{\partial r} + V_h + j\omega\mu_t \Psi + \omega\mu_g \Pi = C_2(r)\,. \tag{17}$$

However, since field recovery via (4) and (5) is obtained via purely transverse differential operators, the terms $C_1(r)$ and $C_2(r)$ do not influence the field calculations. Thus, $C_1(r)$ and $C_2(r)$ may be set equal to zero without loss of generality. Therefore, the following three relations prevail via the scalarization of Faraday's law

$$\nabla_t^2 \Theta - J_{hr} - j\omega\mu_r H_r = 0\,, \tag{18}$$

$$\frac{\partial \Theta}{\partial r} + U_h + j\omega\mu_t \Pi - \omega\mu_g \Psi = 0\,, \tag{19}$$

$$E_r - \frac{\partial \Phi}{\partial r} + V_h + j\omega\mu_t \Psi + \omega\mu_g \Pi = 0\,. \tag{20}$$

Using duality, the following three relations prevail via the scalarization of Ampere's law

$$\nabla_t^2 \Psi + J_{er} + j\omega\varepsilon_r E_r = 0\,, \tag{21}$$

$$\frac{\partial \Psi}{\partial r} - U_e - j\omega\varepsilon_t \Phi + \omega\varepsilon_g \Theta = 0\,, \tag{22}$$

$$H_r - \frac{\partial \Pi}{\partial r} - V_e - j\omega\varepsilon_t \Theta - \omega\varepsilon_g \Phi = 0\,. \tag{23}$$

The first coupled equation for the scalar potential fields Ψ and Θ is obtained via substitution of (21), (22), and (19) into (20), producing the relation

$$\begin{aligned} &\frac{1}{r}\frac{\partial^2 \Psi}{\partial r^2} + \frac{\varepsilon_t}{r\varepsilon_r}\nabla_t^2 \Psi + \frac{k_h^2}{r}\Psi + \frac{\omega\varepsilon_t(\tau_e + \tau_h)}{r}\frac{\partial \Theta}{\partial r} = \\ &-\frac{\varepsilon_t}{r\varepsilon_r}J_{er} + \frac{1}{r}\frac{\partial U_e}{\partial r} + \frac{j\omega\varepsilon_t}{r}V_h - \frac{\omega\varepsilon_t\tau_h}{r}U_h, \end{aligned} \tag{24}$$

where $\tau_e = \varepsilon_g / \varepsilon_t$, $\tau_h = \mu_g / \mu_t$, and $k_h^2 = k_t^2(1-\tau_h^2)$, with $k_t^2 = \omega^2 \varepsilon_t \mu_t$. Using the relations

$$\frac{1}{r}\frac{\partial^2 \Psi}{\partial r^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r}\left(\frac{\Psi}{r} \right) \right], \tag{25}$$

$$\frac{1}{r}\frac{\partial \Theta}{\partial r} = \frac{\partial}{\partial r}\left(\frac{\Theta}{r} \right) + \frac{1}{r}\left(\frac{\Theta}{r} \right), \tag{26}$$

$$\frac{1}{r}\frac{\partial U_e}{\partial r} = \frac{\partial}{\partial r}\left(\frac{U_e}{r} \right) + \frac{1}{r}\left(\frac{U_e}{r} \right), \tag{27}$$

equation (24) becomes

$$\begin{aligned} &\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial \bar{\Psi}}{\partial r} \right] + \frac{\varepsilon_t}{\varepsilon_r}\nabla_t^2 \bar{\Psi} + k_h^2 \bar{\Psi} \\ &+ \omega\varepsilon_t(\tau_e + \tau_h)\left[\frac{\partial \bar{\Theta}}{\partial r} + \frac{\bar{\Theta}}{r} \right] = -\frac{\varepsilon_t}{\varepsilon_r}\bar{J}_{er} \\ &+ \frac{\partial \bar{U}_e}{\partial r} + \frac{\bar{U}_e}{r} + j\omega\varepsilon_t \bar{V}_h - \omega\varepsilon_t\tau_h \bar{U}_h, \end{aligned} \tag{28}$$

where $\bar{\Psi} = \Psi / r$, $\bar{\Theta} = \Theta / r$, $\bar{J}_{er} = J_{er} / r$, $\bar{U}_e = U_e / r$, $\bar{U}_h = U_h / r$, and $\bar{V}_h = V_h / r$. Using duality, the second coupled equation for the scalar potential fields Ψ and Θ is identified as

$$\begin{aligned} &\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial \bar{\Theta}}{\partial r} \right] + \frac{\mu_t}{\mu_r}\nabla_t^2 \bar{\Theta} + k_e^2 \bar{\Theta} \\ &- \omega\mu_t(\tau_e + \tau_h)\left[\frac{\partial \bar{\Psi}}{\partial r} + \frac{\bar{\Psi}}{r} \right] = \frac{\mu_t}{\mu_r}\bar{J}_{hr} \\ &- \frac{\partial \bar{U}_h}{\partial r} - \frac{\bar{U}_h}{r} + j\omega\mu_t \bar{V}_e - \omega\mu_t\tau_e \bar{U}_e, \end{aligned} \tag{29}$$

where $k_e^2 = k_t^2(1-\tau_e^2)$, $\bar{J}_{hr} = J_{hr} / r$, and $\bar{V}_e = V_e / r$.

In summary, the steps of the scalar potential formulation are to first solve (28) and (29) for the scalar potentials $\bar{\Psi}$ and $\bar{\Theta}$ (and hence $\Psi = r\bar{\Psi}$ and $\Theta = r\bar{\Theta}$). Next, the scalar potentials Φ and Π are computed via (22) and (19), respectively. The radial fields E_r and H_r are computed via (21) and (18), and the transverse fields are computed based upon (4) and (5). Enforcement of boundary conditions for a given problem finalize the scalar potential solution procedure.

3. Salient Remarks

Several important comments regarding the scalar potential formulation are now made. There are three primary advantages to the scalar potential formulation. First, the vector Maxwell equations are scalarized, which leads to mathematical simplification. Second, physical insight is enhanced due to the decomposition into radial, transverse lamellar, and transverse rotational components, which aids in the visualization of field behavior. Third, sources are naturally accommodated, which are typically absent in other formulations [4]. It is noted that the scalar potential formulation can be extended to include bianisotropic gyrotropic media. The primary drawback is that the scalar potential formulation appears to be incapable of handling materials more general than gyrotropic media.

It is observed in (21) and (18), the radial electric and magnetic fields E_r and H_r are maintained by Ψ and Θ, respectively. Due to the coupling of these potentials, as evidenced by (28) and (29), only hybrid modes (i.e., modes that are neither TE^r nor TM^r) exist in a spherical gyrotropic medium. However, modes that are TE^r and TM^r can exist independently only if a given problem is radially invariant or if $\tau_e = -\tau_h$, but this is not realistic from a practical standpoint.

A uniaxial scalar potential formulation in spherical coordinates is easily identified. In this case, the gyrotropic parameters ε_g and μ_g are zero, leading to two independent formulations, one that is TE^r and one that is TM^r. The TE^r (i.e., $E_r = 0$) formulation is summarized as

$$\nabla_t^2\Theta - J_{hr} - j\omega\mu_r H_r = 0\,, \tag{30}$$

$$\frac{\partial\Theta}{\partial r} + U_h + j\omega\mu_t\Pi = 0\,, \tag{31}$$

$$H_r - \frac{\partial\Pi}{\partial r} - V_e - j\omega\varepsilon_t\Theta = 0\,, \tag{32}$$

$$\begin{aligned}&\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\bar{\Theta}}{\partial r}\right] + \frac{\mu_t}{\mu_r}\nabla_t^2\bar{\Theta} + k_t^2\bar{\Theta} = \\ &\frac{\mu_t}{\mu_r}\bar{J}_{hr} - \frac{\partial\bar{U}_h}{\partial r} - \frac{\bar{U}_h}{r} + j\omega\mu_t\bar{V}_e,\end{aligned} \tag{33}$$

$$\vec{E}_t = \nabla_t\times(\hat{r}\Theta)\,, \tag{34}$$

$$\vec{H}_t = \nabla_t\Pi\,. \tag{35}$$

Physically, this result reveals that a radial and transverse lamellar magnetic current density maintains a radial and transverse magnetic field, and a transverse rotational electric current density maintains transverse rotational electric field, as anticipated from Maxwell's equations. The TM^r formulation is easily obtained via duality and is omitted for brevity.

In the case of isotropic media, the scalar potential formulation for the case of TE^r (and TM^r via duality) is readily identified via (30)-(35) through the simplification $\varepsilon_r = \varepsilon_t = \varepsilon$ and $\mu_r = \mu_t = \mu$, resulting in

$$\nabla_t^2\Theta - J_{hr} - j\omega\mu H_r = 0\,, \tag{36}$$

$$\frac{\partial\Theta}{\partial r} + U_h + j\omega\mu\Pi = 0\,, \tag{37}$$

$$H_r - \frac{\partial\Pi}{\partial r} - V_e - j\omega\varepsilon\Theta = 0\,, \tag{38}$$

$$\nabla^2\bar{\Theta} + k^2\bar{\Theta} = \bar{J}_{hr} - \frac{\partial\bar{U}_h}{\partial r} - \frac{\bar{U}_h}{r} + j\omega\mu\bar{V}_e, \tag{39}$$

$$\vec{E}_t = \nabla_t\times(\hat{r}\Theta)\,, \tag{40}$$

$$\vec{H}_t = \nabla_t\Pi\,. \tag{41}$$

In the case of a source-free region, this result further simplifies to (upon rearrangement)

$$H_r = \frac{1}{j\omega\mu}\nabla_t^2\Theta\,, \tag{42}$$

$$\Pi = -\frac{1}{j\omega\mu}\frac{\partial\Theta}{\partial r}\,, \tag{43}$$

$$H_r = \frac{\partial\Pi}{\partial r} + j\omega\varepsilon\Theta = -\frac{1}{j\omega\mu}\frac{\partial^2\Theta}{\partial r^2} + j\omega\varepsilon\Theta\,, \tag{44}$$

$$\nabla^2\bar{\Theta} + k^2\bar{\Theta} = 0, \tag{45}$$

$$\vec{E}_t = \nabla_t\times(\hat{r}\Theta) = -\hat{r}\times\nabla_t\Theta\,, \tag{46}$$

$$\vec{H}_t = \nabla_t\Pi\,. \tag{47}$$

Upon defining $\bar{\Theta}$ in terms of another potential F_r, that is

$$\bar{\Theta} = \frac{\Theta}{r} = -\frac{F_r}{r\varepsilon} \quad or \quad \Theta = -\frac{F_r}{\varepsilon}\,, \tag{48}$$

the above formulation can be summarized as

$$\nabla^2\left(\frac{F_r}{r}\right) + k^2\left(\frac{F_r}{r}\right) = 0, \tag{49}$$

$$H_r = \frac{1}{j\omega\varepsilon\mu}\left(\frac{\partial^2 F_r}{\partial r^2} + k^2 F_r\right), \tag{50}$$

$$H_r = -\frac{1}{j\omega\varepsilon\mu}\nabla_t^2 F_r\,, \tag{51}$$

$$\Pi = \frac{1}{j\omega\varepsilon\mu}\frac{\partial F_r}{\partial r}\,, \tag{52}$$

$$\vec{E}_t = \frac{1}{\varepsilon}\hat{r}\times\nabla_t F_r\,, \tag{53}$$

$$\vec{H}_t = \frac{1}{j\omega\varepsilon\mu}\nabla_t\frac{\partial F_r}{\partial r}\,. \tag{54}$$

This is the well-known electric vector potential formulation for $\vec{F} = \hat{r}F_r$ in spherical coordinates for isotropic media that is found in standard textbooks in electromagnetics [5]. Duality leads to the standard magnetic vector potential formulation for $\vec{A} = \hat{r}A_r$, which is omitted here for brevity.

4. Conclusion

Manufacturing capabilities (e.g, 3D printing) allows evermore exotic material to be fabricated in various geometries, including spherical coordinates. A scalar potential formulation capable of accommodating spherical anisotropic gyrotropic media with sources present was developed. Advantages and limitations of the technique were discussed. The scalar potentials for analyzing subclasses of gyrotropic media (i.e., uniaxial and isotropic media) was discussed. In the case of isotropic media, the formulation reduces to the well-known vector potentials taught in standard electromagnetic books. A possible application would be to bias a ferrite with rare earth magnetics laid out in a radially-symmetric grid. This could be of use in EMC and EMI shielding scenarios involving spherical materials and cavities.

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