A wave mode guided by an impedance junction

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Abstract

In this work we reconsider a recent theoretical analysis of wave modes travelling along an impedance junction. We use the Sommerfeld-Malyuzhinets technique as well, but taking into account a well-knwon property of the guided modes and its consequence for the spectrum vector. After a careful study of the poles in the so-called forbidden strip and their removal, we arrive at a Barkhausen-type oscillation condition; from its phase condition it turns out that such modes can exist only along a junction between an inductive impedance surface and a capacitve one. The asymptotic expansion of the exact solution reveals that such modes are closely clung to the junction and the radial extent of their concentration is directly related to the eigenvalue of the problem.

1 Introduction

In a recent paper [1], localised modes supported by an impedance junction has benn studied by the Sommerfeld-Malyuzhinets technique, with the exact solution [2] to diffraction of a skew incident plane-wave by such an obstacle as its point of departure. The reader is referred to the papers cited by [1] for earlier work on this kind of modes.

Here, we revisit the problem by employing the same technique, but taking into account our previous work ([3], [4]) and, in particular, recent publications by Lyalinov ([5], [6]).

2 Summary of Vaccaro's solution

The obstacle under study, an infinite two-part impedance surface, coincides with the yz-plane of a Cartesian coordinate system (x,y,z). The surface impedance of the upper ($y>0$) and lower ($y<0$) part of the plane are denoted by $Z^{\pm}$. In general, Z^{+} differs from Z^{-}, resulting in a jump of the surface impedance at the z-axis. Using the associated circular-cylindrial co-ordinates (r,φ,z), the upper and lower half of the plane are given by $\varphi=\pm\pi/2$.

Now a plane wave impinges non-perpendicularly on the z-axis (the impedance junction) with each of its components, save for an amplitude, varying in the circular cylindrical co-ordinates as $\exp\{\mathrm{i}k[z\cos\beta - r\cos(\varphi-\varphi_0)]\}$. The waves oscillate with time t according to $\exp(-\mathrm{i}\omega t)$. As usual, this factor is dropped.

For the present study, β will be complex-valued with $0\leq \mathrm{Re}\,\beta\leq\pi/2$, in line with [2]. For φ_0 plays no role in this work, we assume simply real-valued φ_0 with $|\varphi_0|\leq\pi/2$.

The z-components, like all other ones of the total field, are expressed in terms of an longitudinal propagator and transversal patterns

$$\begin{bmatrix} E_z(r,\varphi,z) \\ H_z(r,\varphi,z) \end{bmatrix} = \mathrm{e}^{\mathrm{i}kz\cos\beta} \begin{bmatrix} \mathscr{E}(r,\varphi) \\ \mathscr{H}(r,\varphi) \end{bmatrix},$$

with the transversal patterns represented by means of the Sommerfeld integrals

$$\begin{bmatrix} \mathscr{E}(r,\varphi) \\ \mathscr{H}(r,\varphi) \end{bmatrix} = \frac{1}{2\pi\mathrm{i}} \int_{\gamma} \overline{F}_z(\alpha+\varphi)\,\mathrm{e}^{-\mathrm{i}kr\sin\beta\cos\alpha}\,\mathrm{d}\alpha, \quad (1)$$

with the spectrum vector

$$\overline{F}_z(\alpha) = \begin{bmatrix} E_z(\alpha) \\ H_z(\alpha) \end{bmatrix}$$

and γ stands for the Sommerfeld double-loops to be specified later in Subsec 3.6.

In our study, no incident wave is present. In this case, the exact, closed-form solution becomes, see eqn (47) in [2],

$$\overline{F}_z(\alpha) = \left[\overline{\overline{S}}(\alpha)\right]^{-1} \overline{\overline{\Psi}}(\alpha)\,\overline{b}, \quad (2)$$

with the matrices given by

$$\overline{\overline{S}}(\alpha) = \begin{bmatrix} \sin\alpha & -\cos\alpha\cos\beta \\ \cos\alpha\cos\beta & \sin\alpha \end{bmatrix},$$

$\overline{\overline{\Psi}}(\alpha)=\mathrm{diag}[\Psi_{\mathrm{e}}(\alpha),\Psi_{\mathrm{h}}(\alpha)]$ and the column vector $\overline{b}$ to be determined below. The entries of $\overline{\overline{\Psi}}(\alpha)$ are represented in terms of Malyuzhinets function $\psi_{\pi/2}(\alpha)$:

$$\begin{aligned} \Psi_{\mathrm{e,h}}(\alpha) \quad &= \quad \psi_{\pi/2}(\alpha+\pi-\vartheta^{+}_{\mathrm{e,h}})\,\psi_{\pi/2}(\alpha+\vartheta^{+}_{\mathrm{e,h}}) \\ &\quad \times\psi_{\pi/2}(\alpha-\vartheta^{-}_{\mathrm{e,h}})\,\psi_{\pi/2}(\alpha-\pi+\vartheta^{-}_{\mathrm{e,h}}). \quad (3) \end{aligned}$$

In the above expressions, $\vartheta^{\pm}_{\mathrm{e,h}}$ are the Brewster angles defined as $\sin\vartheta^{\pm}_{\mathrm{e}} = Z_0/(Z^{\pm}\sin\beta), \sin\vartheta^{\pm}_{\mathrm{h}} = Z^{\pm}/(Z_0\sin\beta)$. Z_0 denotes the intrinsic impedance of the medium filling the half-space $x>0$.

3 Wave modes travelling along the impedance junction

3.1 Remarks on $E_z(r,\varphi,z), H_z(r,\varphi,z)$ and β

For wave modes moving along a lossless, homogeneous guiding structure, it turns out from the Poynting theorem that the longitudinal components of the electromagnetic field are of either co- or anti-phase. Hence, they, also their transversal patterns $\mathscr{E}(r,\varphi)$ and $\mathscr{H}(r,\varphi)$, can be assumed to be real-valued without sacrificing the generality.

Therefore, it suffices fully to deal with real-valued $\mathscr{E}(r,\varphi)$ and $\mathscr{H}(r,\varphi)$. Before moving on, we turn our attention to the angle β. Especially, we look at the far-field behaviour of the transversal patterns of the axial components. Evaluating the respective Sommerfeld integrals for $kr \gg 1$ with the saddle-point method, as to be shown in Subsec 3.6, the axial components vary with r according to $\exp(\mathrm{i}kr\sin\beta)/\sqrt{r}$.

For the transversal patterns and, hence the axial components, to be purely real-valued and the respective wave modes to be clung to the axis, β must be purely imaginary with $\beta = \mathrm{i}\,\beta_\mathrm{i}$ and $\beta_\mathrm{i} > 0$. It is worth mentioning that studies on diffraction of an incident surface wave by an impedance wedge in the acoustic case [7] and the electromagnetic case [4] show clearly that such a concentrated wave along the edge of the wedge can then and only then be excited if the respective β has those properties.

3.2 A property of the spectrum vector

The material in this subsection is due to M.A. Lyalinov in his email to the author on October 4th 2023.

As shown in Subsec 3.6, the Sommerfeld double-loops run along $\gamma_+ : (\pi+\mathrm{i}\infty, \pi+\mathrm{i}d) \cup (\pi+\mathrm{i}d, -\pi+\mathrm{i}d) \cup (-\pi+\mathrm{i}d, -\pi+\mathrm{i}\infty)$ and γ_- is the image of γ_+ with respect to the origin of the complex α-plane. The positive constant d is chosen in such a way that the spectrum vector is free of poles inside $\gamma_\pm$. See, for example, [7] and [4].

By virtue of this particular shape of γ, (1) is expressible as

$$\begin{bmatrix} \mathscr{E}(r,\varphi) \\ \mathscr{H}(r,\varphi) \end{bmatrix} = \frac{1}{2\pi\mathrm{i}} \int_{\gamma_+} \Big[\overline{F}_z(\alpha+\varphi)\, \mathrm{e}^{kr\sinh\beta_\mathrm{i}\cos\alpha}\, \mathrm{d}\alpha - \overline{F}_z(\alpha^*+\varphi)\, \mathrm{e}^{kr\sinh\beta_\mathrm{i}\cos\alpha^*}\, \mathrm{d}\alpha^* \Big], \qquad (4)$$

where α^* denotes the complex conjugate of α.

To ensure that the left-hand side of (4) be real-valued, there must be

$$\overline{F}_z(\alpha^*) = \left[\overline{F}_z(\alpha)\right]^*. \qquad (5)$$

It means that on the whole real axis of the complex α-plane, the spectrum vector is real-valued.

3.3 Admissible ranges of the Brewster angles and the skewness measure

Inserting (2) into (5) leads to the conclusion that

$$\overline{\overline{\Psi}}(\alpha)\overline{b} = \begin{bmatrix} b_\mathrm{e}\Psi_\mathrm{e}(\alpha) \\ b_\mathrm{h}\Psi_\mathrm{h}(\alpha) \end{bmatrix}$$

must possess the same property as the spectrum vector does. $b_{\mathrm{e,h}}$ are the entries of vector $\overline{b}$ and do not alter with α.

Hence, it is necessary that (for brevity, we drop for the present the indices e and h)

$$\mathrm{Im}\,[b\Psi(p)] = |b\Psi(p)|\sin[\arg b + \arg\Psi(p)] = 0$$

for any real variable p with $-\infty < p < +\infty$, implying that $\arg\Psi(p)$ can never be a function of p.

In order to study $\arg\Psi(p)$ in detail, we make use of (3), the expression of Malyuzhinets function and $\vartheta^\pm = \mathrm{Re}\,\vartheta^\pm + \mathrm{iRe}\,\vartheta^\pm$, and arrive at

$$\arg\Psi(p) = -\int_0^{+\infty} \frac{N(p,\vartheta^+,\vartheta^-,t)}{t\cosh\left(\frac{\pi}{2}t\right)\sinh(\pi t)}\mathrm{d}t,$$

with

$$\begin{aligned} N(p,\vartheta^+,\vartheta^-,t) &= \cosh\left[\left(p+\tfrac{\pi}{2}\right)t\right] \\ &\times \sinh\left[\left(\mathrm{Re}\,\vartheta^+ - \tfrac{\pi}{2}\right)t\right]\sin\left[(\mathrm{Im}\,\vartheta^+)t\right] \\ &+ \cosh\left[\left(p-\tfrac{\pi}{2}\right)t\right] \\ &\times \sinh\left[\left(\mathrm{Re}\,\vartheta^- - \tfrac{\pi}{2}\right)t\right]\sin\left[(\mathrm{Im}\,\vartheta^-)t\right]. \end{aligned}$$

The above expression indicates clearly that $\arg\Psi(p)$ can then and only then be independent of p if the two conditions are simultaneously met:

1. $\mathrm{Re}\,\vartheta^+ = \frac{\pi}{2}$ or $\mathrm{Im}\,\vartheta^+ = 0$,
2. $\mathrm{Re}\,\vartheta^- = \frac{\pi}{2}$ or $\mathrm{Im}\,\vartheta^- = 0$.

Furthermore, there is evidently $\arg\Psi(p) = 0$ and $\arg b = \ell\pi$ with ℓ an arbitrary integer, or what is the same, b is real-valued.

Now let uns consider the admissable range of β as a function of the surface impedance of one half-plane, hence we drop the superindex $\pm$ for this part. It is convenient to introduce the normalsed impedance $\eta = Z/Z_0$ and recall that both η and β are purely imaginary, that is $\eta = \mathrm{i}\eta_\mathrm{i}$ and $\beta = \mathrm{i}\beta_\mathrm{i}$ with β_i positive.

For one and the same impedance surface, the Brewster angles ϑ_e and ϑ_h depend upon η and β: $\sin\vartheta_\mathrm{e} = 1/(\eta\sin\beta) = -1/(\eta_\mathrm{i}\sinh\beta_\mathrm{i}), \sin\vartheta_\mathrm{h} = \eta/(\sin\beta) = \eta_\mathrm{i}/(\sinh\beta_\mathrm{i})$.

For $\mathrm{Re}\,\vartheta_\mathrm{e} = \pi/2$ amounts to $-1/(\eta_\mathrm{i}\sinh\beta_i) \geq 1$, it is only possible for $\eta_\mathrm{i} < 0$, that is an inductive surface when $\sinh\beta_\mathrm{i} \leq -1/\eta_\mathrm{i}$. Similarly, $\mathrm{Re}\,\vartheta_\mathrm{h} = \pi/2$ can be attained for a capacitive impedance ($\eta_\mathrm{i} > 0$) if $\sinh\beta_\mathrm{i} \leq \eta_\mathrm{i}$.

The three eligible cases are listed below:

1. $\mathrm{Re}\,\vartheta_\mathrm{e} = \pi/2$ and $\mathrm{Im}\,\vartheta_\mathrm{h} = 0$: $-1 \leq \eta_\mathrm{i} \leq 0$, $\quad -\eta_\mathrm{i} \leq \sinh\beta_\mathrm{i} \leq -1/\eta_\mathrm{i}$;
2. $\mathrm{Re}\,\vartheta_\mathrm{h} = \pi/2$ and $\mathrm{Im}\,\vartheta_\mathrm{e} = 0$: $\eta_\mathrm{i} \geq 1$, $\quad 1/\eta_\mathrm{i} \leq \sinh\beta_\mathrm{i} \leq \eta_\mathrm{i}$;
3. $\mathrm{Im}\,\vartheta_\mathrm{h} = 0$ and $\mathrm{Im}\,\vartheta_\mathrm{e} = 0$: $\sinh\beta_\mathrm{i} \geq \max\{1/|\eta_\mathrm{i}|, |\eta_\mathrm{i}|\}$.

We remark that the final admissible range of β_i must be the overlapping part of the respective cases for both half-planes.

3.4 Forbidden strip and characteristic equation

Let us assume that spectrum vector $\overline{F}_z(\alpha)$ has a pole α_p. Now we deform γ into its steepest-descent paths $\mathrm{SDP}_{\pm\pi}$: $\mathrm{Re}\,\alpha = \pm\pi$, and this pole can be captured provided $-\pi \leq \mathrm{Re}\,\alpha_\mathrm{p} - \varphi \leq \pi$. The contribution from this pole is proportional to

$$\mathrm{Res}_{a=\alpha_\mathrm{p}}\overline{F}_z(\alpha)\exp(kr\sinh\beta_\mathrm{i}\cos(\alpha_\mathrm{p}-\varphi)).$$

Evidently, the sign of the real part of the exponent is solely determined by $\cos(\mathrm{Re}\,\alpha_\mathrm{p} - \varphi)$ which is non-negative if

$$-\pi/2 \leq \mathrm{Re}\,\alpha_\mathrm{p} - \varphi \leq \pi/2.$$

Because $\varphi \in [-\pi/2, \pi/2]$, we recognise that a pole α_p of the spectrum vector inside the strip $-\pi \leq \mathrm{Re}\,\alpha_\mathrm{p} \leq \pi$ would lead to a (with distance r) non-exponentially decreasing and, therefore, non-physical solution to our problem under study; consequently this strip must be free of any poles of the spectrum vector, hence its name. For details the reader is referred to [5] and [6].

Now it is right time to determine the potential poles of spectrum vector $\overline{F}_z(\alpha)$ in the forbidden strip. As suggested by (2), they are the zeros of $\det\overline{\overline{S}}(\alpha)$ and the poles of $\Psi_{\mathrm{e,h}}(\alpha)$. For the poles of $\psi_\Phi(\alpha)$ which are closest to the origin of the complex α-plane are $\pm(3\pi/2 + 2\Phi)$, hence the respective poles of $\Psi_{\mathrm{e,h}}(\alpha)$ lie well outside the forbidden strip.

The zeros of $\det\overline{\overline{S}}(\alpha)$ are found to be $\alpha_{1,\ell} = \Theta + \ell\pi, \alpha_{2,m} = -\Theta + m\pi$, with $\ell, m = 0, \pm1, \pm2, \ldots$. The auxiliary angle Θ, as shown in [3], is related to β

$$\Theta = \mathrm{i}\ln\cot\frac{\beta}{2} = \frac{\pi}{2} + \mathrm{i}\ln\coth\frac{\beta_\mathrm{i}}{2} = \frac{\pi}{2} + \mathrm{i}\Theta_\mathrm{i}. \tag{6}$$

Four of the zeros are located inside the forbidden strip:

$$\alpha_{1,0} = \Theta = \pi/2 + \mathrm{i}\Theta_\mathrm{i}, \quad \alpha_{1,-1} = \Theta - \pi = -\pi/2 + \mathrm{i}\Theta_\mathrm{i},$$
$$\alpha_{2,1} = -\Theta + \pi = \pi/2 - \mathrm{i}\Theta_\mathrm{i}, \qquad \alpha_{2,0} = -\Theta = -\pi/2 - \mathrm{i}\Theta_\mathrm{i}.$$

How can these four poles of the spectrum vector be removed by means of merely two constants $b_{\mathrm{e,h}}$ at our dispolsal? By realising that two poles in the lower half-plane are the complex conjugates of the two in the upper half-plane and thanks to (5), there are only two independent poles, which are, for convenience, denoted by α_1 for $\alpha_{1,0}$ and α_2 for $\alpha_{1,-1}$. To remove these two poles, there must be

$$\left[\det\overline{\overline{S}}(\alpha)\overline{F}_z(\alpha)\right]_{\alpha=\alpha_{1,2}} = 0,$$

which can be put together in a matrix equation

$$\begin{bmatrix} \mathrm{i}\Psi_\mathrm{e}(\alpha_1) & \Psi_\mathrm{h}(\alpha_1) \\ \mathrm{i}\Psi_\mathrm{e}(\alpha_2) & \Psi_\mathrm{h}(\alpha_2) \end{bmatrix} \begin{bmatrix} b_\mathrm{e} \\ b_\mathrm{h} \end{bmatrix} = 0.$$

For completeness we point out that the zeros of $\Psi(\alpha)$ cannot compensate for $\alpha_{1,2}$.

Hence, the characteristic equation of the present problem reads

$$\frac{\Psi_\mathrm{e}(\alpha_1)}{\Psi_\mathrm{h}(\alpha_1)}\frac{\Psi_\mathrm{h}(\alpha_2)}{\Psi_\mathrm{e}(\alpha_2)} = 1. \tag{7}$$

We take a detour and call the reader's attention to other useful contours of integration for the Sommerfeld integrals (1). One of them is of importance for both asymptotic and numerical evaluation of the Sommerfeld integrals:

$$\begin{bmatrix} \mathscr{E}(r,\varphi) \\ \mathscr{H}(r,\varphi) \end{bmatrix} = \frac{1}{2\pi\mathrm{i}} \int\limits_{\mathrm{SDP}} \overline{F}_z(\alpha+\varphi)\,\mathrm{e}^{kr\sinh\beta_\mathrm{i}\cos\alpha}\,\mathrm{d}\alpha, \tag{8}$$

with the integration to be executed along the steepest-descent paths SDP: $\mathrm{SDP}_{+\pi} \cup \mathrm{SDP}_{-\pi}$.

To deepen our understanding, we divide (7) into an amplitude and a phase condition

$$\left|\frac{\Psi_\mathrm{e}(\alpha_1)}{\Psi_\mathrm{h}(\alpha_1)}\frac{\Psi_\mathrm{h}(\alpha_2)}{\Psi_\mathrm{e}(\alpha_2)}\right| = 1, \tag{9}$$
$$\arg\frac{\Psi_\mathrm{e}(\alpha_1)}{\Psi_\mathrm{h}(\alpha_1)} + \arg\frac{\Psi_\mathrm{h}(\alpha_2)}{\Psi_\mathrm{e}(\alpha_2)} = 2\ell\pi, \quad \ell = 0, \pm1, \pm2, \tag{10}$$

3.5 An implication of the phase condition

Recalling the functional difference-equations to be met by the spectrum vector and realising the validity of

$$\frac{\Psi_\mathrm{e}(\alpha_1^*)}{\Psi_\mathrm{h}(\alpha_1^*)} = \left[\frac{\Psi_\mathrm{e}(\alpha_1)}{\Psi_\mathrm{h}(\alpha_1)}\right]^*$$

for all admissible values of $\vartheta_{\mathrm{e,h}}$, we obtain

$$\arg A_1 = -\mathrm{sgn}(\eta_\mathrm{i}^+)\pi/2. \tag{11}$$

For brevity, we have introduce a short-hand notation $A_1 = |A_1| \exp(\mathrm{i} \arg A_1) = \Psi_{\mathrm{e}}(\alpha_1)/\Psi_{\mathrm{h}}(\alpha_1)$.

Then In a similar fashion we get

$$\arg A_2 = -\mathrm{sgn}(\eta_{\mathrm{i}}^{-})\pi/2,$$

with $A_2 = |A_2| \exp(\mathrm{i} \arg A_2) = \Psi_{\mathrm{h}}(\alpha_2)/\Psi_{\mathrm{e}}(\alpha_2)$.

The phase condition (10) becomes

$$-\pi/2 \left[\mathrm{sgn}(\eta_{\mathrm{i}}^{+}) + \mathrm{sgn}(\eta_{\mathrm{i}}^{-})\right] = 2\ell\pi, \ell = 0, \pm 1, \pm 2, \ldots,$$

permitting solely one solution, namely,

$$\mathrm{sgn}(\eta_{\mathrm{i}}^{+}) + \mathrm{sgn}(\eta_{\mathrm{i}}^{-}) = 0. \tag{12}$$

This proves that the sought-for wave mode can then and only then exist along a junction between an inductive impedance half-plane and a capacitive one.

3.6 Asymptotic analysis of the exact solution

To begin with, we define the Sommerfeld double-loops in the most general case, namely,

- γ consists of two parts γ_+ and γ_-;
- the upper loop is defined as $\gamma_+ : (\pi/2 + \arg k \sin\beta + \varepsilon_1 + \mathrm{i}\infty, \pi/2 + \arg k \sin\beta + \varepsilon_1 + \mathrm{i}d) \cup (\pi/2 + \arg k \sin\beta + \varepsilon_1 + \mathrm{i}d, -3\pi/2 + \arg k \sin\beta + \varepsilon_2 + \mathrm{i}d) \cup (-3\pi/2 + \arg k \sin\beta + \varepsilon_2 + \mathrm{i}d, -3\pi/2 + \arg k \sin\beta + \varepsilon_2 + \mathrm{i}\infty)$;
- the lower loop γ_- is the image of γ_+ with respect to the origin of the complex α-plane;
- the real positive number d is chosen in such a way that none of the loops contains a pole of the spectrum vector;
- the real numbers $\varepsilon_{1,2}$ lie within the interval $(-\pi/2, \pi/2)$.

In a next step we deform γ into its steepest-descent paths $\mathrm{SDP}_{\arg k \sin\beta}(\pm\pi)$: $\mathrm{Re}\,\alpha = \pm\pi - \mathrm{Gd}(\mathrm{Im}\,\alpha, \arg k \sin\beta)$. $\mathrm{Gd}(x,y)$ stands for a generalisation of Gudermann's function $\mathrm{gd}(x)$ used in [8] and [4]:

$$\mathrm{Gd}(x,y) = \arctan \frac{\sinh x \cos y}{1 + \cosh x \sin y}. \tag{13}$$

Further, we note that at the saddle points $\alpha = \pm\pi$, the angle between the $\mathrm{SDP}_{\arg k \sin\beta}(\pm\pi)$ and the real axis of the α-plane equals $\frac{3\pi}{4} - \frac{1}{2} \arg k \sin\beta$. Therefore, the contributions from the saddle points are

$$\left[\overline{F}_z(-\pi+\varphi) - \overline{F}_z(\pi+\varphi)\right] \frac{\mathrm{e}^{\mathrm{i}kr\sin\beta}}{\sqrt{2\pi |k \sin\beta| r}} \mathrm{e}^{\mathrm{i}(\frac{\pi}{4} - \frac{1}{2}\arg k \sin\beta)},$$

providing exactly the r-dependent part used in Subsec 3.1.

Lastly, we arrive at the uniform asymptotic expression for the total field of the wave modes

$$\left[\overline{F}_z(-\pi+\varphi) - \overline{F}_z(\pi+\varphi)\right] \frac{\mathrm{e}^{-kr\sinh\beta_{\mathrm{i}}}}{\sqrt{2\pi k \sinh\beta_{\mathrm{i}} r}}. \tag{14}$$

Evidently, the field is confined inside a circle of radius $1/(k \sinh\beta_{\mathrm{i}})$, dictated by the eigenvalue β_{i} to be found by solving the amplitude condition of oscillation (9). Details will be presented at the meeting.

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