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Extending Quasilinear Theory for Particles Near 90°

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Abstract

Energetic electrons in the radiation belt are trapped when pitch angles are outside the loss cone. Losses and acceleration are well accounted for when considering particles with pitch angles in central values. However, current models rely on a quasilinear theory that does not explain the observable loss rates of particles with pitch angles near 90 degrees. Here we will highlight a derivation of diffusion coefficients under the weak turbulence approximation before suggesting and analysing a potential nonlinear extension to quasilinear theory known as resonance broadening.

1. Introduction

The Earth's radiation belts are two toroidal regions surrounding the planet discovered over 60 years ago, consisting of energetic charged particles. The outer radiation belt is made up predominantly of high energy electrons in the energy range of tens of keV to several MeV between shells of approximately $L = 3$ and $L = 7$. The basic structure of these radiation belts is shown in Fig. 1.

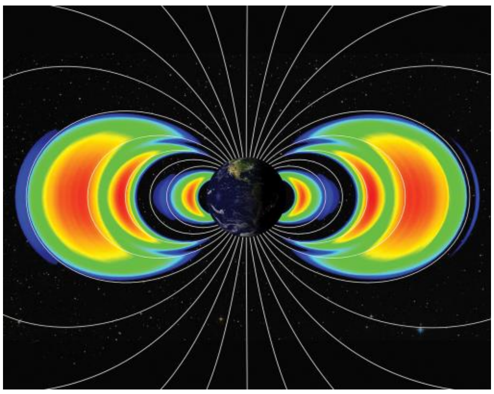

Figure 1. A simplified 2-D cross section of the Earth's radiation belts, representing both the inner and outer radiation belts, and a slot region in between. This image shows the radiation belts during intense solar activity, as a third belt has been formed. Credit: NASA's Goddard Space Flight Center and Johns Hopkins University, Applied Physics Laboratory.

Within the radiation belts, particles are generally trapped within their magnetic field lines. The theoretical upper limit of stably trapped particle flux has been set by Kennel and Petschek [1]. One reason for this limit is dependence on losses to the atmosphere by particle pitch-angle scattering due to interactions with magnetospheric plasma waves. The interaction of particles with plasma waves in the radiation belt is a topic with much left to be discovered. Studies have shown that loss rates in theory do not always line up with observations, often due to not having a full picture of all loss processes [2]. A well-known problem with our current understanding of these interactions is that the behaviour of particles in motion with pitch angles close to 90° from the magnetic field is currently unexplained. While both electromagnetic ion cyclotron (EMIC) waves and chorus waves are known to cause electron dropouts at relativistic energies [3], neither can explain these large pitch-angle loss rates. Realising the true interactions is important for discovering the upper limit on the energy of radiation belt electron flux, which will help with designing shielding for electronic components on spacecraft [4].

2. Background, Theory and Motivation

The dynamics of electrons in the radiation belts can be defined by superposing three distinctive nearly-periodic movements, each working on its own distinctive timescale [5]. Firstly, as a moving charge, each electron moving in the Earth's magnetic field experiences the Lorentz force perpendicular to the field and the particle's velocity. This leads to gyromotion of the particle as it follows the magnetic field line. The magnetic moment of this motion is an adiabatic invariant, wherein it is constant if changes in the system are sufficiently slow. The violation of this invariance through external factors, such as wave-particle interactions, leads to the diffusion of particle energies and pitch angles, hence particle loss or acceleration [6]. Local diffusion of radiation belt particles consists of pitch angle and energy diffusions, in which particles can be accelerated or lost due primarily to interactions with ELF/VLF plasma waves. In the last two decades it has been found that this internal local acceleration can be dominant over the global diffusion with regards to the acceleration of electrons in the radiation belts during and shortly after storm-time events [7].

Wave-particle interactions play a vital role within the radiation belts. These interactions cause particle loss through pitch angle scattering into the loss cone, leading the particles to precipitate into the atmosphere [8]. When plasma waves pass a charged particle, the Doppler-shifted wave may be at the cyclotron frequency of the particle or at a harmonic. This results in a gyroresonant interaction that induces electron momentum diffusion and pitch angle diffusion. In the outer radiation belt this interaction occurs commonly between electrons and very low frequency (VLF) whistler-mode chorus waves. This is because both electrons and chorus waves rotate in a right-handed circular direction. Perfect resonance occurs when

$$\omega - k_{\parallel} v_{\parallel} = n\Omega \qquad (1),$$

where ω is the wave frequency, $k_{\parallel}$ is the wave vector parallel to the magnetic field, $v_{\parallel}$ is the particle's parallel velocity, n is an integer, and Ω is the relativistic cyclotron (Larmor) frequency of the charged particle. When considering waves propagating parallel to the magnetic field, $n = 1$ corresponds to EMIC waves [9] and $n = -1$ corresponds to whistler-mode waves [10]. If oblique waves are considered, a superposition of multiple n values will need to be calculated. The interactions between waves and particles are necessary for calculating accurate local diffusion coefficients. The time and spatial evolutions of the electric and magnetic fields of the waves can affect the charged particles greatly during the time that they are able to interact. However, the exact way these interactions occur is as yet unknown and improvements to the theory need to be explored in future work.

Currently, radiation belt models that include resonant wave-particle interactions rely on the commonly adopted quasilinear theory (QLT) [6, 11, 10]. This theory involves slow changes in the particle distribution function (PDF), f, to provide a good estimate of the behaviour of particles. This theoretical framework makes some approximations that make modelling easier than through including nonlinear effects. Waves are assumed to be broadband with small amplitudes to create a stochastic trajectory. QLT has a restricted domain of applicability, such as large wave amplitudes leading to a possible overestimation in diffusion coefficients. In addition, non-linear effects have been shown to occur often enough to be more significant than previously thought, such as the regularity of very high amplitude whistler-mode waves during disturbed periods [12]. Theoretical approaches to integrating non-linear effects are challenging as it is more difficult to create statistical models for individual significant events. One problem that QLT presents is that the models cannot account for particle diffusion when the equilibrium pitch angle is close to 90° [13]. The distribution of radiation belt electron flux typically peaks near 90°, hence understanding how electrons near this pitch angle can be transported through phase space is important.

3. Weak Turbulence via Markov Approach

When considering wave-particle interactions in the radiation belts, there have been varying approaches to the derivation of the diffusion coefficients. One thorough approach in deriving a theory for both weak turbulence and quasi-linear diffusion is outlined in Allanson et al. [10]. In this section, we will be showing an abridged explanation of Allanson's equations that have been independently rederived by the lead-author. The approach builds on work by Lemons [14], wherein a stationary and time-invariant magnetic field is considered. Allanson et al. [10] utilise a Markovian analysis beginning from the exact relativistic equations of motion, in which particles interact with electromagnetic waves. Beginning from a generalised Fokker-Planck equation for local changes in the radiation belts,

$$\begin{aligned}\frac{\partial f}{\partial t} = &-\frac{1}{G_1}\frac{\partial}{\partial E}(G_1\mathcal{C}_E f) - \frac{1}{G_2}\frac{\partial}{\partial \alpha}(G_2\mathcal{C}_\alpha f) \\ &+\frac{1}{G_1}\frac{\partial^2}{\partial E^2}(G_1\mathcal{D}_{EE} f) \\ &+\frac{1}{G_2}\frac{\partial^2}{\partial \alpha^2}(G_2\mathcal{D}_{\alpha\alpha} f) \\ &+\frac{2}{G_1 G_2}\frac{\partial^2}{\partial\alpha\partial E}(G_1 G_2\mathcal{D}_{\alpha E} f)\end{aligned} \qquad (2),$$

with $G_1 = (E + E_R)\sqrt{E(E + 2E_R)}$ and $G_2 = \sin\alpha$, Allanson et al. [10] shows it is also possible to derive a reduced transport equation for the evolution of a relativistic particle distribution function.

This equation includes particle kinetic energy E, particle rest mass energy E_R , and the drift and diffusion coefficients that are defined as

$$\{\mathcal{C}_\alpha, \mathcal{C}_E\} = \left\{\frac{\langle\Delta\alpha\rangle}{\Delta t}, \frac{\langle\Delta \mathrm{E}\rangle}{\Delta t}\right\} \qquad (4),$$

$$\{\mathcal{D}_{\alpha\alpha}, \mathcal{D}_{\alpha E}, \mathcal{D}_{EE}\} = \left\{\frac{\langle(\Delta\alpha)^2\rangle}{\Delta t}, \frac{\langle\Delta\alpha\Delta \mathrm{E}\rangle}{\Delta t}, \frac{\langle(\Delta E)^2\rangle}{\Delta t}\right\} \qquad (5).$$

Updating the formalism is restrictive as it absorbs drift coefficients within the diffusion coefficients to simplify the equation significantly. The Δt values are chosen to be a suitable timescale in which diffusion or drift can be considered and is used to define the $\Delta\alpha$ and ΔE values.

We begin by considering a uniform background magnetic field $\boldsymbol{B_0} = (B_0, 0,0)$, where x is the parallel direction, leaving the yz plane as perpendicular. For the interaction of many waves with particles either Fourier series or transforms are required for summing over all possible wave-modes. Hence magnetic wave fields for field-aligned waves are defined as

$$B_y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{B}\left(\bar{k}\right)\cos\psi\left(\bar{k},x,t\right)\mathrm{d}\bar{k} \quad (6),$$

$$B_z(x,t) = \mp\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{B}\left(\bar{k}\right)\sin\psi\left(\bar{k},x,t\right)\mathrm{d}\bar{k} \quad (7).$$

Right/left-handed waves are represented by the $\mp$ sign and the phase is $\psi(\bar{k},x,t) = \mathbf{k}\cdot\mathbf{x} - \omega(\bar{k})\Delta t = \bar{k}|\Omega_0|c^{-1}x - \omega(\bar{k})\Delta t$. $\bar{k}$ is a dimensionless variable defined as $\mathbf{k} = \bar{k}|\Omega_0|c^{-1}\hat{\mathbf{x}}$, with relativistic gyrofrequency $\Omega_0 = qB_0/(m_0\gamma)$. If we start with the Lorentz force, we can derive exact relativistic equations of motion for particle position, x, gyrophase, ϕ, pitch angle, α, and kinetic energy, E, due to interactions with these newly introduced parallel field-aligned electromagnetic wave spectra defined in Eqs 6 and 7. An example equation of motion is

$$\frac{dE}{dt} = \mp\sqrt{E(E+2E_R)}\sin\alpha \int_{-\infty}^{\infty}\frac{1}{\eta(\bar{k})}\epsilon(\bar{k})\sin\zeta(\bar{k},x,t)\,d\bar{k} \quad (8),$$

where we define the refractive index of the $\bar{k}$-mode $\eta(\bar{k}) = |\mathbf{k}|c/\omega(\bar{k})$, a mix of wave phase and particle gyrophase, $\zeta(\bar{k},x,t) = \psi(\bar{k},x,t) \pm \phi(x,t)$, and the normalised magnitude of magnetic wave field, $\epsilon(\bar{k}) = \tilde{B}(\bar{k})/B_0$.

The definitions of the diffusion coefficients are shown in equation 5. To obtain these values, we must take the ensemble average and make some additional assumptions. The length of the system is assumed as $L \to \infty$. In addition, the gyrophase of the system's particles are uniformly distributed to zeroth order in the range $[0,2\pi]$. This is the 'random-phase' approximation used often in QLT for spatial and gyrotropic initial symmetry of the system. We now assume that $\epsilon(\bar{k})$ is small, in an assumption known as 'weak turbulence'. For nonlinear understanding we solve the equations of motion up to second order in $\epsilon(\bar{k})$, where

$$E(t) \approx E^{(0)}(t) + E^{(1)}(t) + E^{(2)}(t) \quad (9).$$

The superscript shows the power to which ϵ is taken. We can assume the same form as Eqn 9 for x, E, α and ϕ. Considering the aim is to substitute our dimensions into equation 5, we obtain

$$\Delta E \approx E^{(1)}(t_0+\Delta t) + E^{(2)}(t_0+\Delta t),$$

leading to

$$\{\mathcal{D}_{\alpha\alpha}, \mathcal{D}_{\alpha E}, \mathcal{D}_{EE}\} = \left\{\frac{\langle {\alpha^{(1)}}^2\rangle}{\Delta t}, \frac{\langle \alpha^{(1)}E^{(1)}\rangle}{\Delta t}, \frac{\langle {E^{(1)}}^2\rangle}{\Delta t}\right\} \quad (10).$$

We can then finally obtain our diffusion coefficients, where we use pitch angle diffusion as an example:

$$\mathcal{D}_{\alpha\alpha} = \frac{|\Omega_0|}{4\pi}\frac{c}{|\Omega_0|}\int_{-\infty}^{\infty}\left[\lim_{L\to\infty}\left(\frac{1}{L}\epsilon^2(\bar{k})\right)\right]\left(1 - \frac{\omega(\bar{k})}{\mathbf{k}\cdot\mathbf{v}_\parallel^{(\mathbf{0})}}\cos^2\alpha_0\right)^2 A d\bar{k} \quad (11).$$

Eqn 11 introduces us to a time-dependent factor,

$$A = \frac{[1-\cos(R\Delta t\Omega_0)]}{R^2} \quad (12),$$

with

$$R = \frac{\left(\omega(\bar{k}) - \mathbf{k}\cdot\mathbf{v}_\parallel^{(\mathbf{0})}\right)}{\Omega_0} \pm 1 \quad (13).$$

The A value in Eqn 12 is a function that varies with time or how far the wave-particle interaction is from resonance (R). Under many gyroperiods – a common assumption for quasilinear theory – we find that $A \to \pi\delta(R)$, where this Kronecker delta allows only perfect resonance to lead to diffusion. Taking this theory without assuming long time periods would allow for resonant interactions at a range of frequencies surrounding perfect resonance, with certainly non-negligible amplitudes seen at timescales of $|\Omega_0|\Delta t = 10$. In ref [13], Cai et al. suggest using the pitch angle diffusion coefficient from Summers [11] but replacing a delta function with the resonance broadening function, $f(x)$,

$$f(x) = \begin{cases} \frac{1}{2\Delta_r} & \text{if} \quad |x| \le \Delta_r \\ 0 & \text{if} \quad |x| > \Delta_r \end{cases} \quad (14).$$

Here, Δ_r is the resonance broadening width defined originally as $2\omega_b$, where $\omega_b = \sqrt{kv_\perp\Omega_w/\gamma}$ and $\Omega_w = eB_w/mc$. However, setting $\Delta_r = 4\omega_b$ provided a closer match to test-particle code. Cai et al. [13] believe this is due to the original broadening function including only a single wave interacting with the particles in the system. The updated value is unjustified beyond being a good match for the code, hence a thorough derivation of the theory will be useful in understanding why this is the case. It is our belief that a link between the A value in Allanson et al. [10] and $f(x)$ in Cai et al. [13] exists and may lead to a derivation of resonance broadening derived straight from the equations of motion. To produce data for future studies, we will be using a modification of test-particle simulation code originally written by Donglai Ma (https://github.com/donglai96/taiparticle-uniform) to reproduce pitch angle diffusion graphs presented in Tao et al. [15]. With this code, we can simulate electrons interacting with varying numbers of small amplitude, broadband, incoherent waves in a homogeneous background magnetic field. The wave spectrum is of constant power between $0.2\Omega_e$ and $0.4\Omega_e$ and a mean squared amplitude of 10pT. Initial energy is set so that the waves and particles are at perfect resonance. The results here line up well with expected results using diffusion coefficients in ref [6]. An example of this is shown in Fig. 3, where we can see that the electrons diffuse in a way that produces a linear gradient in $\langle \Delta\alpha^2 \rangle$ and hence a clear diffusion relationship.

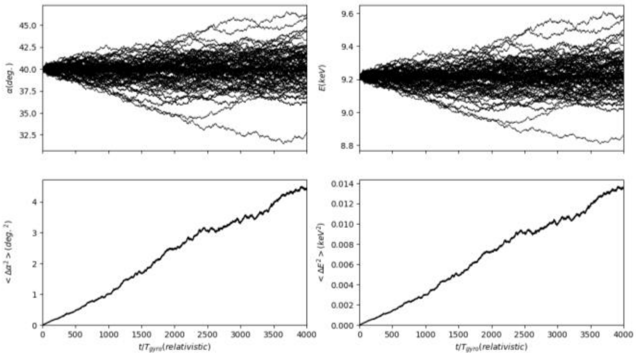

Fig 2. A test particle simulation with 120 electrons initialising at uniformly distributed gyrophase with 40 degree pitch-angles and interacting with 100 whistler-mode chorus waves over 4000 gyroperiods. (top) Evolution of pitch-angle and energy. (bottom) Values of $\langle \Delta\alpha^2 \rangle$ and $\langle \Delta E^2 \rangle$ over time, the gradient of which is used to calculate diffusion coefficients.

4. Conclusion

Development of links between known weak turbulence and resonance broadening diffusion coefficients could be a large step in understanding particle dynamics near 90° and hence the particle flux energy distribution in the radiation belts. Utilising test particle code to explore the wave-particle interactions will provide the means to test variations of these theories. In order to reach theories for resonance broadening from the ground up, a connection may be possible to create between the weak turbulence diffusion coefficients in Allanson et al. [10] and the resonance broadening coefficients in Cai et al. [13].

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