Analytic solutions of the Helmholtz equation with a non-constant wave number

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"*Radio (noun): 1a. The transmission and reception of radio-frequency electromagnetic waves...*" [1].

In most cases 'Radio' is about long distance propagation, and wave propagation is described by the Helmholtz equation. Long distance wave propagation is difficult to study by numerical methods. This stimulates studies of problems in canonical domains, e.g. wedges, cones, etc. Another approach to the Helmholtz equation employs asymptotic methods, such as the ray method. However, the ray method approximates a second order Helmholtz equation by a first order transport equation, which inevitably causes errors. Here we discuss an extension of the ray method which delivers exact analytic solutions of the Helmholtz equation with a non-constant wavenumber.

The solution of the Helmholtz equation $\nabla^2\phi(x) + \Omega^2 L(x)\phi(x) = 0, \quad \Omega = \text{const}$, is sought in the form

$$\phi(x) = u_+(x)\mathrm{e}^{\mathrm{i}\Omega S(x)} + u_-(x)\mathrm{e}^{-\mathrm{i}\Omega S(x)}, \qquad |\nabla S(x)|^2 = L(x), \tag{1}$$

where the eikonal $S(x)$ and amplitudes $u_\pm(x)$ obey the eikonal and transport equations

$$\begin{aligned} &\frac{\mathrm{i}}{2\Omega}\nabla^2 u_+(x) - g(x)e(x)\nabla u_+(x) - b(x)u_+(x) = 0, \qquad g(x) = \pm\sqrt{L(x)}, \qquad e(x) = \frac{\nabla S(x)}{g(x)}, \\ &\frac{\mathrm{i}}{2\Omega}\nabla^2 u_-(x) + g(x)e(x)\nabla u_-(x) + b(x)u_-(x) = 0, \qquad b(x) = \frac{1}{2}\operatorname{div}[g(x)e(x)]. \end{aligned} \tag{2}$$

If $L(x)$ is analytic, then $g(x) = \pm\sqrt{L(x)}$ can be treated as a single-valued function of the two-sheets Riemannian manifold $\mathfrak{G}$ of $\sqrt{L(x)}$, and two equations from (2) can be combined into a single equation on the manifold $\mathfrak{G}$

$$\frac{\mathrm{i}}{2\Omega}\nabla^2 U(\chi) - g(\chi)e(\chi)\nabla U(\chi) - \frac{1}{2}\operatorname{div}[g(\chi)e(\chi)]\,U(\chi) = 0, \quad \chi \in \mathfrak{G}. \tag{3}$$

Equations of this type admit closed-form solutions by the probabilistic Feynman-Kac formula [2, 3]

$$\text{FK-formula:} \quad U(\chi) = \left\langle U(\xi_\tau)\exp\left(-\frac{1}{2}\int_0^\tau g'(\xi_t)\mathrm{d}t\right)\right\rangle, \qquad \text{Ito SDE:} \quad \mathrm{d}\xi_t = \sqrt{\frac{\mathrm{i}}{\Omega}}\,\mathrm{d}w_t - g(\xi_t)\mathrm{d}t, \tag{4}$$

which involves the averaging $\langle\cdot\rangle$ over trajectories of the Ito's stochastic differential equation

The outlined approach to the Helmholtz equation with non-constant wave number combines the physical clarity of the ray method with the versatility of direct numerical methods. This approach starts similarly to the ray method, but instead of looking for approximate solutions of the full transport equations we obtain their exact solutions by the Feynman-Kac formula on a Riemannian manifold, which involves mathematical expectations of certain some functionals depending on the trajectories of the Brownian motion. Unlike the method of path integrals, the proposed method is convenient for numerical simulations. In particular, its implementations require little computer memory and are perfectly suited for parallel computing. It is remarkable that the exact solutions provided by the proposed method appear as straightforward improvements of the asymptotic solutions delivered of the ray method.

References

[1] "Radio (noun)", *Oxford English Dictionary*, https://www.oed.com/dictionary/radio_n?tl=true.

[2] M. Freidlin. *Functional Integration and Partial Differential Equations*. Number 109 in The Annals of Mathematics Studies. Princeton University Press, Princeton, New Jersey, 1985.

[3] E. B. Dynkin. *Markov processes*. Number 121,122 in Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen. Springer-Verlag, Berlin, 1965.