Solving Electromagnetic Wave Equations with Time Varying Characteristics Curves

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Abstract

When multiple temporal boundaries exist in a temporal system, the characteristic curves of the associated wave equations vary with time. We show that these types of problems can be readily addressed by dividing the entire time frame into several durations and repeatedly applying the d'Alembert formula.

1. Introduction

In recent years, time-varying materials have gained increasing interest because they provide an extra degree of freedom to tailor the electromagnetic response of materials in the time domain [1]. Many interesting concepts and applications such as homogenization, cloaks, aiming, photonic band gaps, polarization conversion, and quarter-wave transformers, have been studied from the perspective of temporal boundaries [1, 2, 3, 4]. All these research topics can be categorized as temporal boundary value problems (TBVPs) [5]. In general, TBVPs consider an abrupt change of medium properties that occurs in an infinite and homogeneous space.

In this paper, we study TBVPs from a mathematical perspective, by consider the governing partial differential equations together with the appropriate set of temporal boundary conditions. Here, a quasi-linear method is proposed to solve the representative time-varying wave equations. Using the method of characteristics [6], an analytical transient solution to the wave equations is found. Moreover, the characteristic curves of such wave equations are shown to vary with time in accordance with the medium properties. Finally, the method is verified by considering the example of an antireflection temporal coating (ATC).

2. Analytical Method

To begin, we point out that electromagnetic (EM) problems that employ traditional boundary conditions and temporal boundary conditions fall into two different categories: boundary value problems (BVP) and initial value problems (IVP), respectively. As we know, a boundary condition dictates the behavior of a function on the boundary of its area of definition. An initial condition is similar to a boundary condition, but the so called 'boundary' exists in time.

However, the solution of a BVP and an IVP could be very different, where the latter has a travelling wave solution. Suppose we consider the one-dimensional (1D) scalar wave equation: $u_{tt}(x,t) = c^2 u_{xx}(x,t)$, and employ the method of characteristics. The curves in the $x-t$ plane:

$$\begin{cases} x - ct = C_1 \\ x + ct = C_2 \end{cases} \tag{1}$$

are called the characteristics of the wave equation, where C_1 and C_2 are defined by the initial conditions at $t = 0$ ($i.e., u(x,0)$ and $u_t(x,0)$), and c is the wave speed. Under these conditions, the particular solution can be determined by d'Alembert formula:

$$u(x,t) = \frac{1}{2}[u(x-ct,0) + u(x+ct,0)] + \frac{1}{2c}\int_{x-ct}^{x+ct} u_t(\tau,0)d\tau \tag{2}$$

where $u(x,t)$ is the unknown wave distribution to be solved.

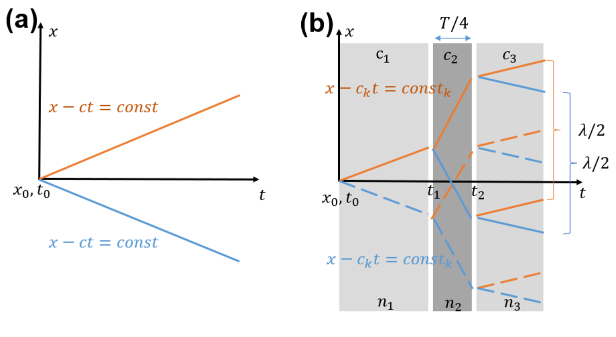

Figure 1. The space-time diagram with characteristic lines in (a) a time-invariant medim, and (b) a temporal system where there is an antireflection temporal coating (two temporal boundaries exist at $\boldsymbol{t_1}$ and $\boldsymbol{t_2}$). The orange and blue lines correspond to the forward and backward waves, respectively.

Now let us consider a more complicated scenario in EM theory. IVPs with multiple temporal boundaries, which correspond to the following quasi-linear wave equation:

$$\frac{\partial^2 \boldsymbol{D}}{\partial t^2} = c^2(t)\nabla^2 \boldsymbol{D}, \tag{3}$$

where $c^2(t) = 1/\mu\varepsilon(t)$. If the medium is time-invariant, then Eq. (3) reduces to a conventional linear wave equation, which can be solved using Eq. (2).

However, if the medium (therefore the wave speed) is time dependent, Eq. (3) becomes quasi-linear. As shown in Fig. 1 (b), the characteristics curve of the wave equation varies with time. We found that this type of problem can be effectively solved by dividing the entire timeframe into several shorter durations, and during each of them, the wave speed can be assumed to remain unchanged. The solution can then be found by applying Eq. (2) multiple times for each temporal duration [5].

3. Numerical Example

Next, we applied the proposed method to the analysis of an antireflection temporal coating [7], which represents an interesting application of temporal boundaries, and also falls into the category of TBVPs. An ATC is similar to a traditional quarter-wave transformer, which is a widely used practical device that can minimize the reflection and maximize the transmission through a multi-layer material system.

As shown in Fig. 1(b), an ATC has an interim state between t_1 and t_2 where the refractive index is n_2. Moreover, we have $\Delta t = t_2 - t_1 = T/4$ and $n_2 = \sqrt{n_1 n_3}$. It is interesting to note that the ATC will minimize not only reflection but also transmission [8].

In this example, the incident waveform is assumed to be a sinusoidal modulated Gaussian wave. We then compute the transient wave distribution using the proposed method.

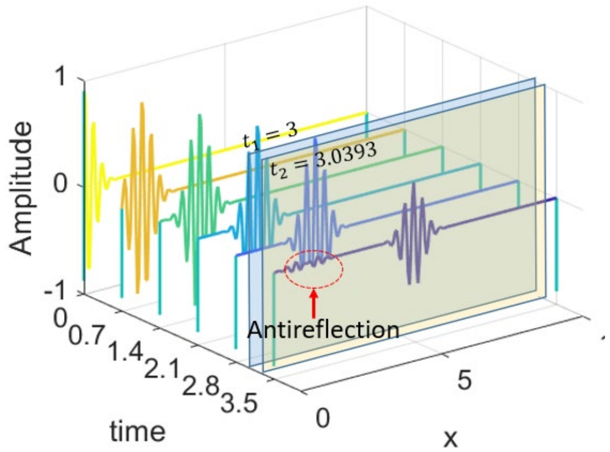

Figure 2 (a) shows the calculated distribution of the D -field as a function of time. The area marked by a red circle indicates that the reflection has indeed been canceled. The underlying reason for the reflection cancellation has been explained in [7] and [8].

4. Conclusion

In summary, we have shown that a temporal system with multiple temporal boundaries can be effectively classified as a type of quasi-linear wave equation problem in which its characteristic curves vary with time. By dividing the entire time-scheme into several shorter durations, it becomes possible to apply the d'Alembert formula repeatedly in each duration. The method is validated by considering the example of an antireflection temporal coating.

6. Acknowledgements

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