3rd URSI AT-AP-RASC, Gran Canaria, 29 May – 3 June 2022

Numerical method for solving hybrid TE-TE wave propagation problems in a shielded plane nonlinear waveguide

Valeria Martynova*(1), Eugene Smolkin(2)
(1) Department of Mathematics and Supercomputing, Penza State University, Krasnaya Str. 40, Penza, Russia, 440026, https://pnzgu.ru
(2) University of Gävle, Kungsbäcksvägen 47,Gävle 801 76 Sweden, https://www.hig.se

Abstract

The problem of hybrid TE-TE wave propagation in a plane nonlinear anisotropic shielded waveguide is considered. This problem is reduced to a new type of nonlinear two-parameter problems with one spectral parameter. For the determination of approximate solutions of the problem a version of the shooting method specifically developed for this type of two-parameter problems is proposed. Numerical results are presented.

1 Introduction

The study of eigenmodes of nonlinear cylindrical and planar waveguide structures [1, 2, 3, 4, 5, 6] was resulting in emergence of new problem statements and mathematical methods. In particular, a new class of two-parameter nonlinear problems is of interest. In such problems, only one of the parameters is spectral, and the second is selected taking into account some condition. This class includes problems on the propagation of symmetric hybrid TE-TM waves in a plane nonlinear waveguide [7, 8] and azimuthal symmetric hybrid TE-TM waves in nonlinear cylindrical waveguide [9, 10].

In this paper, we numerically study the problem of propagation of hybrid TE-TE waves in a shielded plane nonlinear anisotropic waveguide. This problem also belongs to a new class of nonlinear two-parameter problems, which is described above. The propagation constant of the hybrid TE-TE wave is a spectral parameter of the problem, an additional parameter is related to the absolute value of the magnetic field at one of the waveguide boundaries and chosen such that there is a nontrivial solution to the eigenvalue problem.

2 Statement of the problem

Consider the propagation of an electromagnetic wave in a plane dielectric waveguide

$$\Sigma := \left\{(x,y,z) : 0 < x < h, (y,z) \in \mathbb{R}^2\right\},$$

where $h > 0$. The waveguide Σ is located in the Cartesian coordinate system $Oxyz$ (see figure 1). At the boundaries $x = 0, x = h$ the waveguide has perfectly conducted walls.

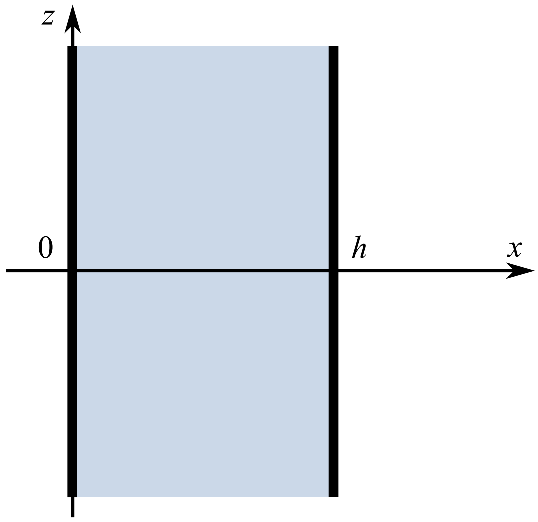

Figure 1. Geometry of the problem.

Let us consider electromagnetic waves $(\mathbf{E},\mathbf{H})e^{-i\omega t}$ propagating in the waveguide Σ, where $\omega > 0$ is a circular frequency; the complex amplitudes $\mathbf{E}$, $\mathbf{H}$ have the form

$$\mathbf{E} = \left(0, \mathrm{E_y(x)e^{i\gamma z}}, \mathrm{E_z(x)e^{i\gamma y}}\right)^\top,$$

$$\mathbf{H} = \left(\mathrm{H_{x1}(x)e^{i\gamma z} + H_{x2}(x)e^{i\gamma y}}, \mathrm{H_y(x)e^{i\gamma y}}, \mathrm{H_z(x)e^{i\gamma z}}\right)^\top.$$

This type of wave is called hybrid TE-TE waves. γ is a propagation constant of a hybrid TE-TE wave.

The permittivity inside the waveguide is described by the diagonal tensor

$$\hat{\varepsilon} = \begin{pmatrix} \star & 0 & 0 \\ 0 & \varepsilon_l + \alpha_1 \mathrm{E}_y^2 + \alpha_2 \mathrm{E}_z^2 & 0 \\ 0 & 0 & \varepsilon_l + \alpha_2 \mathrm{E}_y^2 + \alpha_1 \mathrm{E}_z^2 \end{pmatrix},$$

where $\varepsilon_l > 1$, $\alpha_1, \alpha_2 > 0$ are real constants, $\star$ is a tensor element that does not affect the propagation of hybrid TE-TE waves.

Maxwell's equations for the complex amplitudes $\mathbf{E}$, $\mathbf{H}$ have the form

$$\nabla \times \mathbf{H} = -i\omega\varepsilon_0 \hat{\varepsilon}\mathbf{E}, \quad \nabla \times \mathbf{E} = i\omega\mu_0 \mathbf{H}, \tag{1}$$

where $\varepsilon_0 > 0$ is the permittivity of vacuum, $\mu_0 > 0$ is the permeability of vacuum.

Thus the complex amplitudes **E**, **H** satisfy equations (1), and tangential components of **E** vanish on the perfectly conducted walls ($x = 0$, $x = h$).

Substituting the complex amplitudes into Maxwell's equations (1), we obtain the following system

$$\begin{cases} \mathrm{E}_y'' = \gamma^2 \mathrm{E}_y - k_0^2 \big(\varepsilon_l + \alpha_1 \mathrm{E}_y^2 + \alpha_2 \mathrm{E}_z^2\big) \mathrm{E}_y, \\ \mathrm{E}_z'' = \gamma^2 \mathrm{E}_z - k_0^2 \big(\varepsilon_l + \alpha_2 \mathrm{E}_y^2 + \alpha_1 \mathrm{E}_z^2\big) \mathrm{E}_z, \end{cases} \tag{2}$$

where $k_0^2 = \omega^2 \mu_0 \varepsilon_0$,

$$\mathrm{H}_{x1} = -\gamma(\omega\mu_0)^{-1}\mathrm{E}_y, \qquad \mathrm{H}_y = -(i\omega\mu_0)^{-1}\mathrm{E}_z',$$
$$\mathrm{H}_{x2} = \gamma(\omega\mu_0)^{-1}\mathrm{E}_z, \qquad \mathrm{H}_z = (i\omega\mu_0)^{-1}\mathrm{E}_y'.$$

The wave field in the waveguide Σ can be represented using functions $u_1(x) := \mathrm{E}_y(x)$, $u_2(x) := \mathrm{E}_z(x)$.

From system (2) we get

$$\begin{cases} u_1'' = \gamma^2 u_1 - k_0^2 \big(\varepsilon_l + \alpha_1 u_1^2 + \alpha_2 u_2^2\big) u_1, \\ u_2'' = \gamma^2 u_2 - k_0^2 \big(\varepsilon_l + \alpha_2 u_1^2 + \alpha_1 u_2^2\big) u_2. \end{cases} \tag{3}$$

Taking into account the conditions at the boundaries, we obtain

$$u_1(0) = 0, \qquad u_2(0) = 0, \tag{4}$$
$$u_1'(0) = A_1, \qquad u_2'(0) = A_2, \tag{5}$$
$$u_1(h) = 0, \qquad u_2(h) = 0, \tag{6}$$

where A_1, $A_2 > 0$ are an unknown constants. We assume that

$$A_1^2 + A_2^2 = A^2, \tag{7}$$

where $A > 0$ is a known constant, $A_1, A_2 \in (0, A)$. Thus, we need to find A_1, and A_2 can be found using the following formula

$$A_2 = \sqrt{A^2 - A_1^2}. \tag{8}$$

Definition 1 *The problem P is to find a couple (γ, A_1), where $\gamma \in (-\infty, +\infty)$, $A_1 \in (0, A)$, at which for a given value of $A > 0$ and condition* (7)*, there exist solutions $u_1, u_2 \in C^2[0,h]$ of system* (3)*, satisfying the boundary conditions* (4)–(6)*.*

We stress that the problem P is a two-parameter nonlinear problem, where γ is a spectral parameter, A_1 is an additional parameter, and A_2 is determined by the formula (8).

3 Numerical method

First of all let us consider the Cauchy problem for system (3) with the initial conditions (4)–(5).

Using the «integral» continuity theorem [11], it can be proved that for sufficiently small α_2 the Cauchy problem (3)–(5) is globally uniquely solvable for $x \in [0,h]$ and its (classical) solution $u_1 \equiv u_1(x;\gamma,A_1)$, $u_1 \equiv u_2(x;\gamma,A_2)$ continuously depends on the point $(x;\gamma,A_1) \in [0,h] \times (-\infty,+\infty) \times (0,A)$.

Using the condition on the boundary $x = h$ (6), we can obtain the system of dispersion equations

$$\begin{cases} \Delta_1(\gamma, A_1) \equiv u_1(h;\gamma,A_1) = 0, \\ \Delta_2(\gamma, A_2) \equiv u_2(h;\gamma,A_2) = 0, \end{cases} \tag{9}$$

where quantities $u_1(h;\gamma,A_1)$ and $u_2(h;\gamma,A_2)$ are obtained from the solution of the Cauchy problem (3)–(5).

Introduce a grid

$$\Big\{ \big(\gamma^{(i)}, A_1^{(j)}\big) : \gamma^{(i)} = a_1 + i\tau_1,\ A_1^{(j)} = j\tau_2,$$
$$i = \overline{0,n},\ \tau_1 = \frac{a_2 - a_1}{n},\ j = \overline{1,m-1},\ \tau_2 = \frac{A}{m} \Big\}$$

with steps $\tau_1 > 0$, $\tau_2 > 0$, where a_1, a_2 are real fixed constants. Values of steps τ_1 and τ_2 allow us to control the accuracy of the solution.

Solving the Cauchy problem (3)–(5) for each grid point, we obtain solutions $u_1\big(h;\gamma^{(i)},A_1^{(j)}\big)$ and $u_2\big(h;\gamma^{(i)},A_1^{(j)}\big)$, $i = \overline{0,n}$, $j = \overline{1,m-1}$. Let us remark that the solutions are continuously dependent on parameters γ and A_1. This implies that if

$$\Delta_1\big(\gamma^{(i)},A_1^{(j)}\big)\Delta_1\big(\gamma^{(i)},A_1^{(j+1)}\big) \leqslant 0,$$

then there exists a point $\big(\gamma^{(i)},\widehat{A}_1\big)$, where $\widehat{A}_1 \in \big(A_1^{(j)},A_1^{(j+1)}\big)$, such that $\Delta_1\big(\gamma^{(i)},\widehat{A}_1\big) = 0$. Similarly, if

$$\Delta_1\big(\gamma^{(i)},A_1^{(j)}\big)\Delta_1\big(\gamma^{(i+1)},A_1^{(j)}\big) \leqslant 0,$$

then there exists a point $\big(\widehat{\gamma},A_1^{(j)}\big)$, where $\widehat{\gamma} \in \big(\gamma^{(i)},\gamma^{(i+1)}\big)$, such that $\Delta_1\big(\widehat{\gamma},A_1^{(j)}\big) = 0$. In other words, we can find a set of couples $(\gamma^{(k)},A_1^{(k)})$, where $k = \overline{1,p}$ and p is the number of the determined points. This set is presented as a curve in the plane $OA_1\gamma$ (the blue curve in figure 2).

Applying the same approach to the second equation of (9), we obtain set of approximate solutions of the equation $\Delta_2\big(\alpha,\beta\big) = 0$. This set is presented as another curve in the plane $OA_1\gamma$ (the green curve in figure 2). The dependence $\gamma(A_1)$ is called the dispersion curve.

On the next step, we determine points of intersections of the curves (the red point in figure 2). These points are approximate solutions of system (9). Consequently, the intersection points are solutions of the problem P.

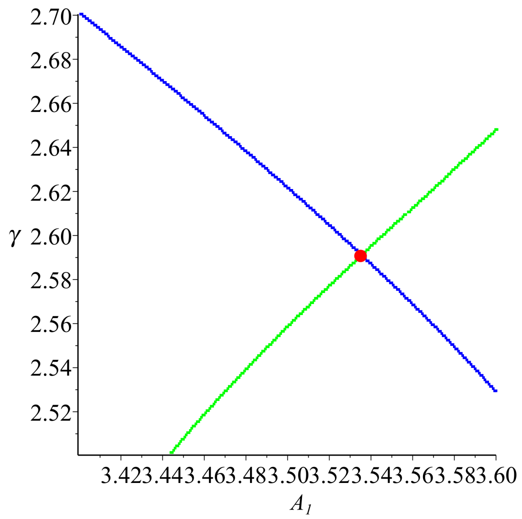

Figure 2. Numerical solution of the system (9).

4 Numerical results

The numerical solutions of problem P shown in figure 3 are obtained using the numerical method described in section 3.

The following values of parameters are used for calculations in figures 3–7: $k_0 = 1$ mm, $\varepsilon_l = 4$, $A = 5$ V/mm, $h = 9$ mm, $\alpha_1 = 0.12$ mm^2/V^2, $\alpha_2 = 0.0002$ mm^2/V^2, $a_1 = 0$, $a_2 = 4$, $\tau_1 = 0.008$, $\tau_2 = 0.01$.

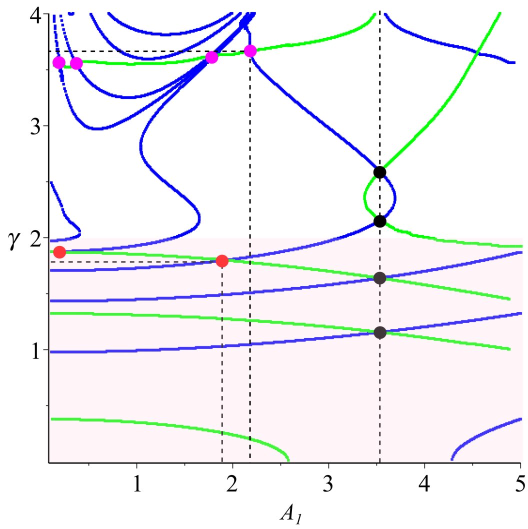

Figure 3. The graph of dispersion curves.

The points of intersection of blue and green curves in figure 3 correspond to approximate solutions of the problem P. Intersection points, for which $A_1 \approx A_2$, are marked with black color. If $\alpha_1 = \alpha_2 = 0$, we obtain linear problem. It can be proved that all couples of linear solutions (γ, A_1) can exist only inside the domain

$$D = \left(0, \sqrt{k_0^2 \varepsilon_l}\right) \times (0, A).$$

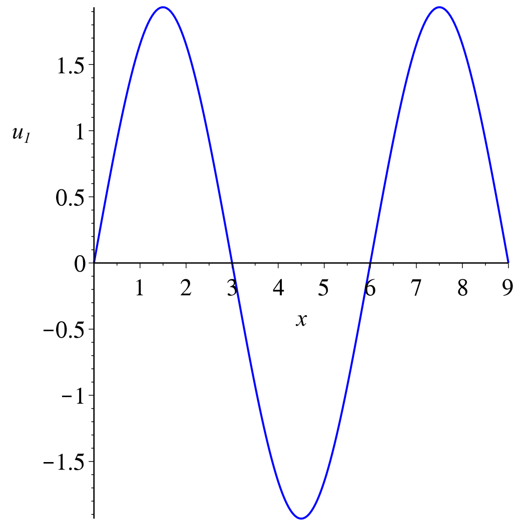

Figure 4. The graph of eigenfunction u_1 for $\gamma = 1.799$ 1/mm, $A_1 = 1.923$ V/mm.

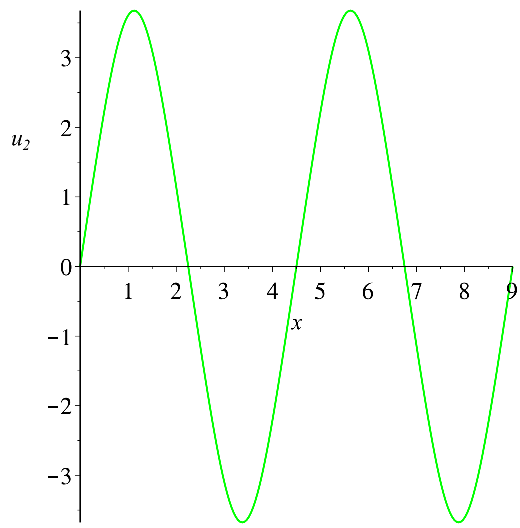

Figure 5. The graph of eigenfunction u_2 for $\gamma = 1.799$ 1/mm, $A_1 = 1.923$ V/mm.

The domain D is highlighted with magenta color in figure 3. The red intersection points are located in the domain D and are close to linear solutions. The purple intersection points are purely nonlinear solutions.

The eigenfunctions u_1, u_2 for the red intersection point $(\gamma, A_1) = (1.799, 1.923)$ and for the purple intersection point $(\gamma, A_1) = (3.668, 2.178)$ marked in figure 3 are presented in figures 4–7. It can be seen that an increase in the value of the propagation constant of a hybrid TE-TE wave leads to an increase in the amplitudes of functions u_1, u_2.

5 Conclusion

The numerical method for solving the problem of propagation of hybrid TE-TE waves in a shielded plane waveguide filled with nonlinear anisotropic medium is proposed. The

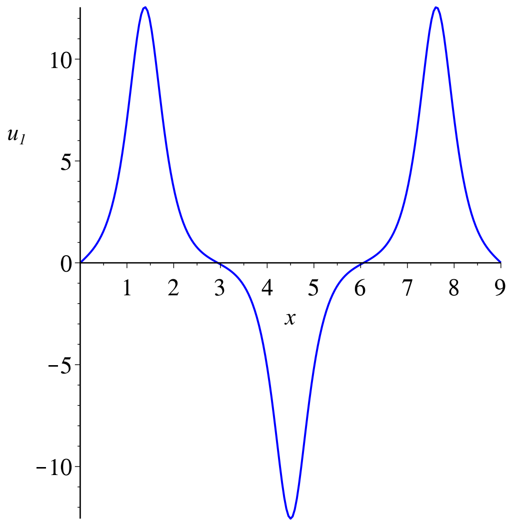

Figure 6. The graph of eigenfunction u_1 for $\gamma = 3.668$ 1/mm, $A_1 = 2.178$ V/mm.

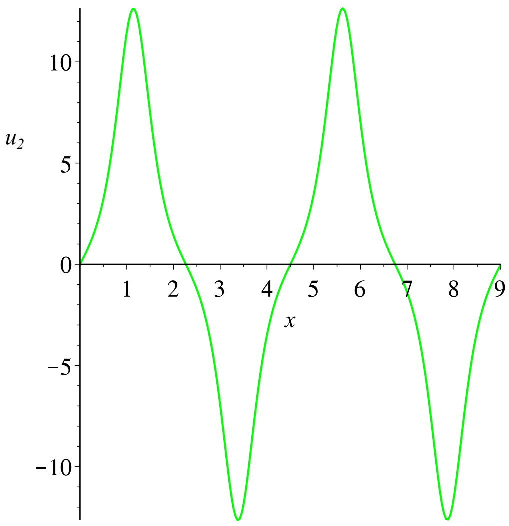

Figure 7. The graph of eigenfunction u_2 for $\gamma = 3.668$ 1/mm, $A_1 = 2.178$ V/mm.

method is based on the solution to the Cauchy problem (3)–(5) and makes it possible to numerical determine propagation constants of a new type of waves, namely hybrid TE-TE waves. In addition, solutions that have no analogues in the linear case are found.

6 Acknowledgements

This work was supported by Russian President Scholarship, project number 2051.2021.5.

References

[1] P. N. Eleonskii, L. G. Oganes'yants, V. P. Silin, "Cylindrical Nonlinear Waveguides," *Soviet Physics JETP*, **35**, 1, 1972, pp. 44–47.

[2] A. D. Boardman, T. Twardowski, "Transverse-electric and transverse-magnetic waves in nonlinear isotropic waveguides," *Physical Review A*, **39**, 5, 1989, pp. 2481–2492, doi: 10.1103/PhysRevA.39.2481.

[3] H. W. Schürmann, V. S. Serov, Yu. V. Shestopalov, "Solutions to the Helmholtz equation for TE-guided waves in a three-layer structure with Kerr-type nonlinearity," *Journal of Physics A: Mathematical and General*, **35**, 50, 2002, pp. 10789–10801, doi: 10.1088/0305-4470/35/50/309.

[4] H. W. Schürmann, Y. Smirnov, Y. Shestopalov, "Propagation of TE waves in cylindrical nonlinear dielectric waveguides," *Physical Review E – Statistical, Nonlinear, and Soft Matter Physics*, **71**, 7, 2005, p. 016614, doi: 10.1103/PHYSREVE.71.016614.

[5] Yu. G. Smirnov, D. V. Valovik, "Nonlinear Effects of Electromagnetic TM Wave Propagation in Anisotropic Layer with Kerr Nonlinearity," *Advances in Mathematical Physics*, **2012**, 2012, p. 609765, doi: 10.1155/2012/609765.

[6] V. Yu. Kurseeva, D. V. Valovik, "Theory of Nonlinear Guided Electromagnetic Waves in a Plane Two-Layered Dielectric Waveguide," *Mathematical Problems in Engineering*, **2017**, 2017, p. 4215685, doi: 10.1155/2017/4215685.

[7] V. Y. Kurseeva, "Electromagnetic Non-Polarized Symmetric Hybrid Wave Propagation in a Plane Waveguide with Nonlinear Anisotropic Permittivity," *Lobachevskii Journal of Mathematics*, **39**, 8, 2018, pp. 1075–1089, doi: 10.1134/S199508021808005X.

[8] Y. Smirnov, E. Smolkin, V. Kurseeva, "The new type of non-polarized symmetric electromagnetic waves in planar nonlinear waveguide," *Applicable Analysis*, **98**, 3, 2019, pp. 483–498, doi: 10.1080/00036811.2017.1395865.

[9] E. Smolkin, "The Azimuthal Symmetric Hybrid Waves in Nonlinear Cylindrical Waveguide," *Progress In Electromagnetics Research Symposium Proceedings*, 2017, pp. 348–353, doi: 10.1109/PIERS.2017.8261763.

[10] Y. Smirnov, E. Smolkin, "On the existence of non-polarized azimuthal-symmetric electromagnetic waves in circular dielectric waveguide filled with nonlinear isotropic homogeneous medium," *Wave Motion*, **77**, 2018, pp. 77–90, doi: 10.1016/J.WAVEMOTI.2017.11.001.

[11] L. S. Pontryagin, "Ordinary Differential Equations," *Pergamon Press*, Addison-Wesley, Reading, MA; Palo Alto, CA; London, 1962.