3rd URSI AT-AP-RASC, Gran Canaria, 29 May – 3 June 2022

Optimization of Inverse Problems involving Surface Reconstruction: Least Squares Application

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Abstract

This article addresses the least-squares method, which is vital in inverse scattering problems involving the reconstruction of inaccessible rough surface profiles from the measured scattered field data. The unknown surface profile is retrieved by a regularized recursive Newton algorithm which is regularized by the Tikhonov method. The importance of the least-squares application reveals at this point, where the unknown surface profile is expressed as a linear combination of some appropriate basis functions. Thus, the problem of obtaining the unknown rough surface is reduced to finding the unknown coefficients of these functions. As an optimization problem, the choice of appropriate basis functions, as well as the number of their expansions for rough surface imaging problems are essential for the iterative solutions. The validation limits and the performances of different basis functions are presented via several numerical examples.

1 Introduction

Imaging of inaccessible rough surface problems are of great interest as they have a wide range of engineering applications areas of remote sensing, geophysics, material and surface science, optics, underwater communication applications, and so on. These problems are classified as the inverse scattering problems, which is considered an engineering optimization problem, where the unknown rough surface profile is retrieved based on the scattered field data measured on a particular domain. Studies in this context include scenarios where inaccessible roughness is either perfectly electric conducting (PEC) (or sound soft boundary in acoustic case) [1–5] or forms a boundary between two penetrable media [6–10]. As the retrieval of the surface profile with scattered field data is an ill-posed nonlinear problem [11], the majority of these studies are based on the regularized-iterative approaches like Newton-based Levenberg-Marquardt algorithm or Landweber method in the sense of least squares [12]. Accordingly, the unknown of the recursive regularized procedure is expanded as a linear combination of suitable basis functions with some unknown coefficients, which acts as a regularization parameter. The choice of appropriate basis function is crucial to observe a successful reconstruction. Different type of basis functions is applied in the open literature depending on the considered roughness scenarios. In this context, [2, 9] use cubic and [4, 10] utilize the quartic spline-type basis functions to reconstruct locally perturbed rough surface profiles. Sine-type and exponential type basis functions are applied in [5, 6] to get the image of unknown local roughness. A recursive algorithm to reconstruct periodic grating profiles is proposed in [3] which applies trigonometric polynomial expansions. Gaussian-type basis functions are recommended and applied to reconstruct moderately rough random rough surface profiles in [8].

In this paper, the reconstruction of PEC rough surface profiles via Newton-type regularized recursive algorithms is presented taking the least squares application into account, in detail. For different roughness scenarios are considered and the performances of various basis functions are analyzed. The outline of the paper is as follows. In section 2, the electromagnetic scenario including the geometry and electromagnetic properties of the problem are briefly introduced. Then, the linearized and regularized iterative inversion algorithm and the least squares application with the selected basis functions explained in detail. Section 4 is devoted to analyze the performances of the basis functions for different inversion scenarios. Concluding remarks follow in Section 5.

2 2D Scattering Problem Definition

As depicted in Fig.1, a locally perturbed PEC rough surface Γ is characterized by a height function $x_2 = f(x_1)$. The

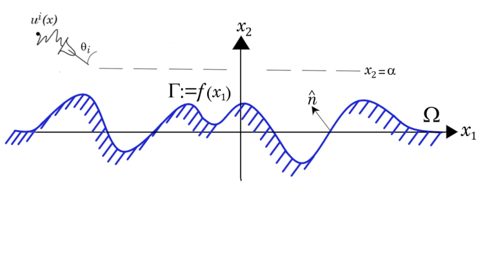

Figure 1. 2D Scattering Scenario

upper half space, denoted as Ω, is free space with the wave number $k = 2\pi/\lambda$, where λ is the wavelength. Let $x = (x_1, x_2)$ and the surface be illuminated by a TE-polarized incident tapered plane wave, $\vec{E}^i = \hat{x}_3 u^i(x)$ with

$$u^i(x) = e^{ik(x_1 \sin\theta_i - x_2 \cos\theta_i)(1+\zeta(x))} \cdot e^{-\left(\frac{x_1 + x_2 \tan\theta_i}{g}\right)^2}, \qquad (1)$$

where θ_i is the angle of incidence, g is the tapered parameter, and $\zeta(x)$ is the angle dependent function, precisely defined in [13]. Regarding the homogeneity in x_3 direction, the problem is reduced to a scalar one. Thus, let u^s denote the scattered field and define the total field upper medium such that $u = u^i + u^s$, then the direct scattering problem aims to find the total field, which satisfies the Helmholtz equation in the upper medium and vanishes on the rough surface, *i.e.,*

$$\left(\Delta + k^2\right) u = 0, \text{ in } \Omega, \tag{2}$$

$$u = 0, \text{ on } \Gamma. \tag{3}$$

Accordingly, the scattered field is an outgoing since it obeys the Sommerfeld radiation condition

$$\lim_{|x|\to\infty} \sqrt{|x|}\left(\frac{\partial u^s}{\partial x} - iku^s\right) = 0,\ x \in \Omega \setminus \Gamma \tag{4}$$

uniformly in all directions. Now, for the case of the inverse scattering problem, the following is considered: given the incident field u^i together with the scattered field u^s for a fixed wavenumber k, it is desired to determine the unknown surface profile $f(x_1)$. The scattered field, which should be collected by some receiver antennas located above the roughness is obtained synthetically by solving the associated direct problem through the spectral approach described in [13].

3 Retrieval of the Rough Surface Profile

In order to determine the unknown surface profile f via the scattered field u^s the measured on the line $\{x_2 = \alpha \mid \alpha > \max\{f\}\}$, the solution of the direct scattering problem is applied, which defines an operator $D: f \to u^s$ that maps f onto the scattered field u^s. Hence, the inverse problem consists in solving $D(\partial u/\partial n, f) = u^s$ for f, which is an improperly posed nonlinear equation. Accordingly, the operator is defined in terms of the surface integral equation:

$$D(\nu, f) = u^s \tag{5}$$

such that

$$D(\nu, f) = -\int_{\Gamma(f)} G(x;y)\nu(y)ds(y)|_{y_2=f(y_1)}, \tag{6}$$

where $G(x;y) = (i/4)H_0^{(1)}(k|x-y|)$ is the fundamental solution of the Helmholtz equation in 2D, and $\nu = \partial u/\partial n$. It has been proved that there exists a solution for these problems but it is also proved that they are nonlinear and severely ill-posed so that they should be linearized as well as regularized [12]. To this aim, let Γ_0 be an approximated rough surface profile characterized by $x_2 = f_0(x_1)$ and ν_0 is the surface density of f_0. Then, (6) is linearized via Newton's method

$$D(\nu, f) \approx D(\nu, f) + D'(\nu_0, f_0)\delta f_0 \tag{7}$$

Here, δf_0 is the updated correlation function such that $f_1 = f_0 + \delta f_0$ for which (7) must be solved. In addition, $D'(\nu_0, f_0)$ is the Fréchet derivative of the operator with respect to the rough surface variation $f(x_1)$ [8], precisely:

$$D'(\nu_0, f_0)\delta f_0 = \int_{\Gamma(f_0)} \frac{\partial G(x;y)}{\partial f}\nu_0 \delta f_0 ds(y)\bigg|_{y_2=f_0}. \tag{8}$$

With the aid of Newton, (7) is a linear but still ill-posed. Thus, the problem is regularized by Tikhonov regularization in least squares sense [14] to stabilize the solution of (7). In regard to least squares method, δf_0 expressed as an expansion of some basis functions ϕ_n with unknown coefficients a_n $(n = 1, 2, \cdots N)$, namely,

$$\delta f_0(x_1) = \sum_{n=1}^{N} a_n \phi_n(x_1). \tag{9}$$

In this context, substituting (9) into (8) gives

$$D'(\nu_0, f_0)\delta f_0 = -\sum_{n=1}^{N} a_n \int_{\Gamma(f_0)} \frac{\partial G_1(x;y)}{\partial f}\nu_0(y)\phi_n(y)ds(y) \tag{10}$$

Accordingly, substituting (10) into (7) yields

$$D(\nu, f) - D(\nu_0, f_0) = -\sum_{n=1}^{N} a_n \int_{\Gamma(f_0)} \frac{\partial G_1(x;y)}{\partial f}\nu_0(y)\phi_n(y)ds(y), \tag{11}$$

which can be written in a more compact form as

$$\sum_{n=1}^{N} C_{mn} a_n = \Delta u_m^s. \tag{12}$$

Here, C_{mn} represents the integral of (11) written on the rough surface which can be evaluated via numerical trapezoidal rule of integration, and Δu_m^s denotes the difference of the scattered field data, *i.e.,* $D(\nu, f) - D(\nu_0, f_0)$ for a set of $x_1^1, x_2^1, \cdots x_M^1$ grid points. Hence, the whole discretized system can be reduced into a matrix notation

$$\bar{\bar{C}}_{M\times N} \times \bar{A}_{N\times 1} = \bar{\Delta u^s}_{M\times 1}, \tag{13}$$

where the number of N unknown coefficients of δf_0 are in the vector $\bar{A}$. Lastly, due to the ill-posed nature of the problem, it is regularized via Tikhonov as

$$\left(\gamma\bar{\bar{I}} + \bar{\bar{C}}^\dagger \cdot \bar{\bar{C}}\right) \cdot \bar{A} = \bar{\bar{C}}^\dagger \cdot \bar{\Delta u^s} \tag{14}$$

In (14), $\bar{\bar{I}}$ is $N \times N$ identity matrix, $\bar{\bar{C}}^\dagger$ represents the ad-joint of $\bar{\bar{C}}$ and γ is the Tikhonov parameter such that $0 < \gamma < 1$. The benefits of the least-squares method's application are about not only ensuring more stable results but also underlying in the dimension of (14). Accordingly, the solution requires the inversion of $N \times N$, where N is the number of basis functions defined in (12). Hence, in the least-squares application, the whole system has no essential dependency to the predetermined range of unknown roughness, and therefore, it also prevents "inverse crime". Besides, another advantage of the least-squares application is that it provides a fast solution. For instance, using discretization, the integral operator stated above can be transformed into matrices. However, as compared to the $N \times N$ system, this direct discretization would contain too many unknowns depending on the length of the surface. In addition, a rough discretization process on integral operators can cause stability problems. Consequently, the determination of expansion number N acts like a regularization parameter. That is, choosing a smaller N than required leads to poor reconstructed surface image quality while choosing N too big leads instabilities due to the ill-posed structure of the inverse problem. The entire procedure outlined in (7)-(14) is iteratively repeated. That is to say, for i^{th} iteration ($i \geq 1$), solving

$$D'(v_i, f_i)\,\delta f_i = u^s(x) - D(v_i, f_i) \tag{15}$$

for δf_i yields a new reconstructed surface profile f_{i+1} such that

$$f_{i+1} = f_i + \delta f_i. \tag{16}$$

It is expected that after each iteration, the reconstructed surface profile should be getting closer to the original one so that the ℓ_2- norm $\|\delta f_i\| < \|\delta f_{i-1}\|$. The iterations are repeated until a predetermined threshold value ξ such that $\|\delta f_i\| \leq \xi$. In the numerical examples given in the next section, $\xi = 5 \times 10^-3$.

In this study, spline-type functions, trigonometric and Gaussian basis functions will be applied as the basis functions in least squares application. To start with the spline-type basis functions, let the total length of the rough surface be $2L$ the parameter $h = 2L(N+5)$, and $t_n = (n+2)(h-L)$. Then the spline basis function $\phi_n^s(t) = \phi^s((t-t_n)/h)$ for $n = 1, 2, \cdots N$, where

$$\phi^s(t) := \sum_{j=0}^{k} \frac{(-1)^j}{k!} \binom{k+1}{j} \left(t + \frac{k+1}{2} - j\right)_+^k \tag{17}$$

with $z_+^k = z^k$ for $z \geq 0$ and $z_+^k = 0$ for $z < 0$ [10]. In the numerical analysis given in the following section, the degree of the spline function is chosen as both cubic ($k = 3$) and quartic ($k = 4$).

Next, Gaussian type spline function, which is proposed in [8], is defined as

$$\phi_n^g(x) = e^{\frac{(x-\mu_n)}{2\rho^2}}, \tag{18}$$

where ρ determines the spacing between the different basis functions that combine to form the model. Finally, if the length of the rough surface is assumed as L, then the trigonometric basis function is defined as [6]

$$\phi_n^t(x) = e^{\frac{i2\pi n x_1}{L}} \tag{19}$$

4 Numerical Examples

The section is reserved to observe the reconstruction performances of aforementioned basis functions. To analyze the results quantitatively, an error is defined as

$$e = 100 \times (\|f - f_r\| / \|f\|) \tag{20}$$

where $\|\cdot\|$ is ℓ_2 norm and the functions f and f_r denotes the actual and reconstructed surface profiles, respectively. To have a general idea, 25 randomly formed rough surfaces, which are generated as result of stationary Gaussian-type random process [8] with parameters RMS height 0.05λ and the correlation length $\ell = 0.95\lambda$ are reconstructed and the average of the error values are obtained as shown in Table.1. According to obtained the smallest error, $e(\%)$, the best option is Spline-type basis functions. It is remarkable that spline functions perform the more successful reconstruction compared to Gaussian type basis functions, even though the surface has Gaussian type random roughness.

Table 1. Parameters for Rough Surface Reconstructions

Surface	Basis Function	N	$e(\%)$
25 Rand. Surf (mean)	$\phi_n^g(x_1)$	25	15.23
Deterministic in (21)	$\phi_n^g(x_1)$	25	9.12
25 Rand. Surf (mean)	$\phi_n^s(x_1)$	22	10.37
Deterministic in (21)	$\phi_n^s(x_1)$	20	5.99
25 Rand. Surf (mean)	$\phi_n^t(x_1)$	25	17.18
Deterministic in (21)	$\phi_n^t(x_1)$	25	5.69

In the second example, it is desired to reconstruct a deterministic surface, namely

$$f(x_1) = 0.3\cos\left(\frac{2}{3}\pi x_1\right)e^{-0.08x_1^2} \tag{21}$$

As the roughness is represented with a cosine type function, the best reconstruction is obtained with trigonometric type basis functions, which is shown in Fig.2. As shown in Table.1 there is a very small difference between the errors obtained with spline and trigonometric functions, Among them, Gaussian type yields the biggest error values.

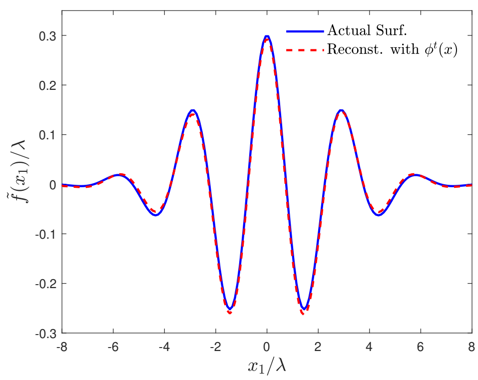

Figure 2. Reconstruction of Deterministic rough surface with trigonometric basis functions ($\phi^t(x)$)

The last reconstruction example is carried for emphasizing the importance of N. The number of basis functions acts like a regularization parameters in reconstructions. Too small and too big selection of N yields insufficient reconstruction performances, which is determined by trial and errors. In this example, a random rough surface with parameters $h = 0.05\lambda$ and $l = 0.95\lambda$ is reconstructed with spline functions for different N numbers. The errors are obtained $e(\%) = 18.032$, $e(\%) = 12.661$ and $e(\%) = 14.271$, for $N = 18$, $N = 22$ and $N = 26$ respectively. The reconstruction for $N = 22$ is shown in Fig.3

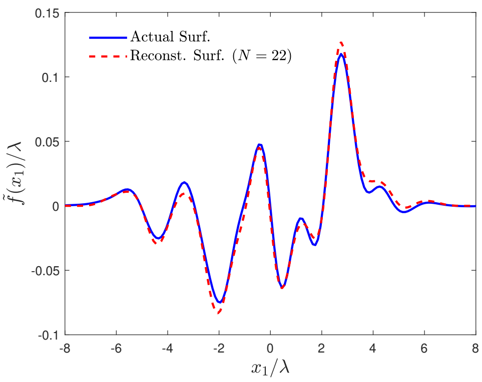

Figure 3. Reconstruction of random rough surface with spline-type basis functions ($\phi^s(x)$)

5 Conclusion

The application of the least-squares method to the inverse-imaging algorithm is analyzed by summarizing the general regularized recursive solutions of the inverse scattering problem covering rough surface imaging. Accordingly, the selection of the appropriate basis function, as well as the expansion number, is crucial since it acts as a regularization parameter of the regularized iterative solution of the inverse-imaging problem. It is shown that among three fundamental basis functions, B-spline functions are the best option for locally perturbed inaccessible rough surface imaging. It is also observed that even if the surface roughness is characterized as a Gaussian type random process, spline-type basis functions still yield better quantitative performance compared with that of Gaussian type basis functions. On the other hand, if the surface distortion is suitable to be modeled as the sine-cosine type, then trigonometric basis functions will also yield successful results.

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