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Three-Dimensional Generalized Sources for Integral Equation Solvers

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Abstract

To enable the development of low-rank compressible generalized sources integral equations, a modified three-dimensional source is presented. Its design is an extension of a two-dimensional generalized source, composed of an elemental source and an auxiliary absorbing shield. On the shield, equivalent sources are defined to approximately cancel the broadside radiation in the direction of potential observers on an essentially convex geometry. The design principles are discussed and means for the efficient computation of the generalized source's modified Green's function, from tabulated samples using non-uniform grids, are described.
Keywords: Integral equations, direct solvers, Green's function, fast methods.

1. Introduction

Algebraic fast integral equation (IE) solvers rely on the fast compression into compact representations of moment matrix blocks that represent the wave interactions between source and observer subdomains of the analyzed geometry. The compression can enable both the fast application of the moment matrix to a vector, which is required for iterative solvers [1]-[3], and the fast computation of its compressed factorized form, as part of a fast direct solver [4]-[8]. The common and convenient approach for compression of blocks is by low-rank (LR) approximation [9]: wave interactions that are of low number of degrees-of-freedom translate into matrix blocks that are of inherently-low ranks. However, only certain IE formulations and geometrical configurations lead to inherently rank-deficient matrix blocks, i.e., blocks of rank that scale significantly slower than the number of basis and testing functions participating in the interaction. Such, for example, are the cases of electrically small geometries [7] or electrically-large ones that are of reduced dimensionality (e.g., elongated [4] or quasi-planar topologies [6]). Interactions between arbitrary surfaces in three dimensions (3D), for example, exhibit ranks proportional to the number of unknowns and are, thus, incompressible in the LR sense.

Recently, it has been shown that enhanced rank-deficiency can be obtained for closed essentially-convex geometries, via modification of the IE [10]-[12]. Two approaches were proposed for two-dimensional (2D) problems: (i) the generalized equivalence [10],[11] and (ii) the generalized sources [12] IE formulations. The latter, which replaces the standard sources of the conventional IE by directional ones producing only weak high-rank broadside interactions, is more geometrically adaptive than the former, which relies on the introduction of an internal scatterer to eliminate the line-of-sight. The directional sources are composed of the original equation's elemental sources and auxiliary ones that approximately cancel their radiation in the direction of broadside observers. This leads to an effective reduction of the interactions' dimensionality and, therefore, to increased rank-deficiency and moment matrix compressibility.

This approach was demonstrated and various challenges in its implementation were addressed in [14]-[17], for two-dimensional IEs. Its extension to three-dimensions relies, first and foremost, on the design of auxiliary sources that produce a deep shadow, and for which the modified Green's function (MGF) can be computed quickly and efficiently. This paper presents a candidate for a generalized source design, based on absorbing auxiliary source shields, that can serve as the cornerstone of a 3-D generalized source IE solver. The approach is demonstrated for the problem of a scalar potential scattering by a soft boundary.

2. Problem formulation

Consider a three dimensional essentially-convex surface S illuminated by a time harmonic scalar incident field $u^{\mathrm{i}}(\boldsymbol{r})$. On S, a soft boundary condition for the total field $u(\boldsymbol{r})$ is satisfied, giving rise to a scattered field $u^{\mathrm{s}}(\boldsymbol{r})$. The fields on S satisfy the surface IE

$$u^{s}(\boldsymbol{r}) = \int_{\boldsymbol{S}} (\partial_{\mathrm{n}} u) G_{\mathrm{f}}(\boldsymbol{r},\boldsymbol{r}') ds' = -u^{\mathrm{i}}(\boldsymbol{r}),\ \boldsymbol{r} \in S \quad (1)$$

In (1), the kernel $G_{\mathrm{f}}(\boldsymbol{r},\boldsymbol{r}') = \exp(-jk|\boldsymbol{r}-\boldsymbol{r}'|)/4\pi|\boldsymbol{r}-\boldsymbol{r}'|$ is the free space Green's function, describing the field at $\boldsymbol{r}$ produced by a point source at $\boldsymbol{r}'$ with k being the wave-number and ∂_{n} denoting the normal derivative. When discretized using the method of moments (MoM), the off-diagonal blocks of the resulting matrix are, generally, not rank-deficient, in the sense that their rank may scale linearly with their smallest dimension. The generalized sources IE approach seeks to replace $G_\mathrm{f}(\boldsymbol{r},\boldsymbol{r}')$ by a MGF $G_\mathrm{m}(\boldsymbol{r},\boldsymbol{r}')$ that produces more rank-deficient off-diagonal blocks. This will be done by extending the concept of absorptive shields, presented in [12] for the 2-D case.

3. Generalized Sources in 3-D

The isotropic sources in (1) are replaced with directional ones that radiate weakly toward the interior of S. To that end, for each source, we add an auxiliary source distribution, such that

$$G_\mathrm{m}(\boldsymbol{r},\boldsymbol{r}') = G_\mathrm{f}(\boldsymbol{r},\boldsymbol{r}') + G_\mathrm{aux}(\boldsymbol{r},\boldsymbol{r}'), \tag{2}$$

where $G_\mathrm{aux}(\boldsymbol{r},\boldsymbol{r}')$ describes the contributions of the auxiliary sources. In [12], these sources comprised windowed equivalent electric and magnetic sources on a circle centered at $\boldsymbol{r}'$ that exactly cancel $G_\mathrm{f}(\boldsymbol{r},\boldsymbol{r}')$. The window produces a smoothed truncation to ensure that all auxiliary sources lie within S. Here, the auxiliary sources field is constructed by combining single and double layer potentials of equivalent source distributions on a sphere, windowed to be entirely included in S. The geometry of the shielding surface S' is illustrated in Fig. 1. The contribution $G_\mathrm{aux}(\boldsymbol{r},\boldsymbol{r}')$ is computed via

$$G_\mathrm{aux}(\boldsymbol{r},\boldsymbol{r}') = \int_{S'} W(\boldsymbol{r}')\left(\frac{\partial G(\boldsymbol{r}')}{\partial n'} G(\boldsymbol{r}-\boldsymbol{r}') - G(\boldsymbol{r}')\frac{\partial G(\boldsymbol{r}-\boldsymbol{r}')}{\partial n'}\right) ds' \tag{3}$$

The window $W(\boldsymbol{r})$ introduces a source amplitude taper, designed to mitigate the diffraction from the edges of the shield, which may otherwise severely limit the depth of the shadow. In the example presented here, the window was defined as

$$W(\boldsymbol{r}) = W(\theta) = \begin{cases} 1 & 0 \le \theta \le \theta_1 \\ \cos^2\left(\frac{\pi(\theta-\theta_1)}{2(\theta_0-\theta_1)}\right) & \theta_1 \le \theta \le \theta_0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

The parameters θ_0 and θ_1 were tuned in order to balance between the depth of the shadow behind the shield and the shadow region's width.

The total field produced by an elemental source and its corresponding shield is presented in Fig. 2. It can be seen

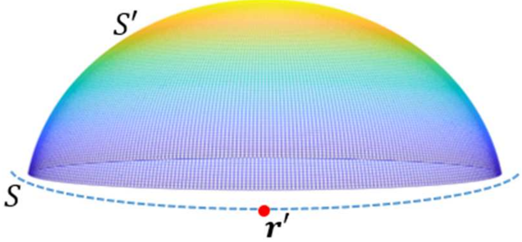

Figure 1. A portion of a spherical shield centered at an elemental source.

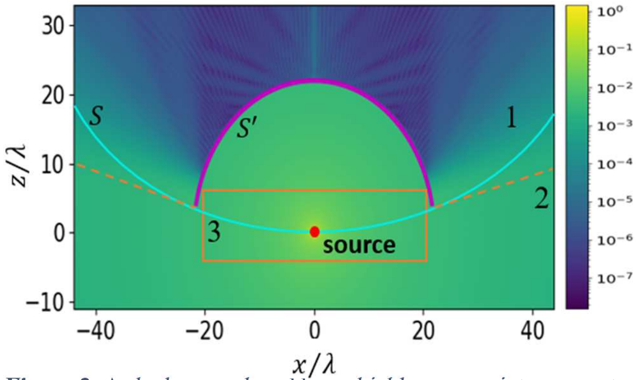

Figure 2. A shadow produced by a shield near a point source at $\boldsymbol{r}' = 0$.

that the field near the source is hardly influenced by the shield. This is important for maintaining a solution space that is similar to that of the original IE, which naturally corresponds to strong and smooth near interactions. Behind the shield a deep shadow is obtained, significantly weakening the source's interaction with that region compared to the sideways interaction, which remains strong, and that with the source's immediate neighborhood. These properties are of importance for low-rank compression of the off-diagonal moment matrix blocks.

Clearly, the computation of the MGF is significantly more expensive than that of G_f. To avoid the potential bottleneck due to the repeated MGF computation, for arbitrary source and observer pairs, a scheme for the reconstruction of the MGF from tabulated samples is used. Thanks to the shield's rotational symmetry, it is sufficient to sample the MGF in the plane. Within the plane, the reconstruction is performed in a region-dependent manner: outside the shield's spherical surface, in the region where the elemental source is strictly obscured by the shield (deep into Region 1 in Fig. 2), the total MGF is sampled using circular non-uniform grids centered near the shield cross-section's edge. The sampling rate is dictated by the effective width of the rim, which is observed to be the source of the residual radiation to this region. Phase compensation is used to enable coarser sampling. For the region where the elemental source is only partially obscured (due to the taper) by the shield (near the border with Region 2 in Fig. 2), the *total* MGF is phase-compensated with respect to the elemental source's location. The reconstruction in both cases is via local polynomial interpolation and phase restoration. In the region beyond the shield's radius that has a line-of-sight to the elemental source (Region 2 in Fig. 2), the *shield* contribution is phase-compensated with respect to the shield's edge. This contribution is interpolated and phase-restored, and then added to the analytically computed elemental source's contribution. Overlap areas between neighboring regions are defined, in which a partition of unity blends smoothly the regions' reconstruction methods, to avoid artifacts that may arise from the discontinuity in the MGF and its derivative in the direction normal to the regions interface. For points within the shield's radius (Region 3 in Fig. 2), the shield's contribution is sampled densely on a regular grid, without phase compensation and restoration.

4. Further Work

The proposed MGF and its fast reconstruction scheme should next be incorporated in an IE solver for the computation of the method of moments matrix and demonstration of the compression of its off-diagonal blocks. It is expected that the shadow depth will dictate the accuracy up to which compression will be obtained, thus continuing to motivate the further optimization and improvement of the shields.

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