Complete Helmholtz Decomposition on Multiply Connected Subdivision Surfaces

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Subdivision surfaces is the tool of choice computer graphics, largely used to represent complex surfaces that are easily refined, deformable and sufficiently smooth. Indeed, the literature on subdivision argues persuasively for using these representations as opposed to non-uniform B-spline (NURBS), largely due to higher order smoothness and flexibility of sub-division surfaces. Indeed, subdivision surfaces can be C^2 almost everywhere. Given the ability to model geometry to high fidelity, the next step has been to use these methods to model physics [1]. Indeed, as shown in [2], it is possible to develop methods that enable iso-goemetric analysis, i.e., use the same basis functions to define both the geometry and the physics on the geometry.

Our approach in developing subdivision based iso-geometric methods for electromagnetic analysis has taken two directions. One wherein we define a Helmholtz decomposition (for simply connected structures). Another is to define div-conforming basis set. In both cases, we demonstrate anticipated convergence rates; note the order of polynomial is fixed in an isogeometric setting. For simply connected structures, the possibility of an exact Helmholtz decomposition is highly beneficial. As was shown in [2], the use of these basis makes it trivial to develop well conditioned systems that has now been refined to analyze electrically large objects [3].

However, a striking deficiency is the inability of to handle multiply connected regions. Specifically, consider a current $\mathbf{J}(\mathbf{r}) = \mathbf{J}^1(\mathbf{r}) + \mathbf{J}^2(\mathbf{r}) + \bar{\omega}(\mathbf{r})$, where $\mathbf{J}_1$, $\mathbf{J}_2$, and $\bar{\omega}(\mathbf{r})$ are divergence free, curl-free, and harmonic components, respectively. On a subdivision surface, the surface of each patch or triangle formed by three control nodes is influenced by nodes belonging 1-ring of patches that surround a patch. This is counter to our conventional Lagrangian picture of a geometry. That said, it is possible to associate an effective basis function, $\xi_i(\mathbf{r})$, with each control vertex i, such that $\psi(\mathbf{r}) \approx \sum_i a_i^1 \xi(\mathbf{r})$. Ditto for $\psi(\mathbf{r})$. Critically, both $\xi_i(\mathbf{r})$ and its gradient go smoothly to zero at the boundary of its support and $\xi_i(\mathbf{r})$ forms a partition of unity [2]. It is then apparent that one can construct a Helmhotz decomposition for simply connected structures based on these basis. Constructing one for the harmonic component in multiply connected structures is a challenge. To overcome this bottleneck, we take a numerical approach to construct basis set for the harmonic components. Let $\mathbf{J}(\mathbf{r}) \approx \sum_n a_n^1 \nabla_s \xi_i(\mathbf{r}) + \sum_n a_n^2 \hat{\mathbf{n}} \times \nabla_s \xi(\mathbf{r}) + \sum_{m=1}^{2g} b_m \bar{h}_m(\mathbf{r})$. In Table 1 where we reconstruct representation of $\mathbf{J}(\mathbf{r}) = \mathbf{X}_\nu(\mathbf{r})$ where $\nu \in \{d, c, h\}$ where the fields are defined analytically on a torus. As is apparent, the norms are close to zero where they should be. The deviation from machine precision is due to geometric approximation of a torus. Details and application to integral equation solvers for electromagnetics will be presented at the meeting.

	$\mathbf{X}_c$	$\mathbf{X}_d$	$\mathbf{X}_h$
$\| \mathbf{J}^1 \|_2$	6.53E-17	5.05E2	3.01E-10
$\| \mathbf{J}^2 \|_2$	5.05E2	6.53E-17	1.67E-18
$\| \bar{\omega}_1 \|_2$	1.09E-12	1.52E-10	3.68E0
$\| \bar{\omega}_2 \|_2$	1.52E-10	-1.33E-12	-3.99E1

Table 1. L^2 norm of each component

References

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