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Direct Inversion of the MLNG Field Evaluation Operator for Fast Solution of Electromagnetic Scattering Problems

Evgeny V. Chernokozhin* and Amir Boag
Tel Aviv University, Tel Aviv 69978, Israel

Abstract

A method for direct inversion of the EFIE operator is proposed. The method is based on the Multilevel Nonuniform Grid (MLNG) approach, which previously has been successfully used in fast iterative solvers. However, the MLNG approach is also suitable for direct solvers. The efficiency of the method stems from the fact that the direct inversion of MLNG-compressed integral operators reduces to the inversion of banded matrices.

1. Introduction

Electromagnetic (EM) scattering problems are often solved using the Method of Moments (MoM). This method has undoubted advantages of versatility and accuracy, but also has serious limitations in frequency and scatterer sizes, since MoM conventionally reduces to a system of linear algebraic equations with a fully populated matrix. The direct solution by the methods based on the Gaussian elimination requires $O(N^3)$ operations, where N is the number of unknowns in the system. "Fast" iterative methods have a computational complexity of $O(N \log N)$ [1] times the number of iterations, which, in the case of slow convergence, can be fairly large.

A significant advantage of direct methods is the possibility to efficiently solve problems with multiple excitations and in the cases of poor convergence of the iterative counterparts. Many efforts have been made to develop "fast" direct methods, more efficient than the traditional Gaussian elimination-based methods. Most of them involve a rank-revealing procedure, aimed at the matrix compression, since the off-diagonal parts of the impedance matrix are often rank deficient. This rank deficiency can be significant when special formulations of the scattering problem are used [2] or under some restrictions on the geometry of the scatterer, e.g., for electrically small [3], strongly elongated [4] or quasi-planar [5] scatterers. As a result, the initial system of linear algebraic equations can be reduced to solving linear systems of significantly lower dimensions. Unfortunately, for electrically large three-dimensional scatterers, no significant low-rank compression can be achieved. To cope with this difficulty, alternative approaches are being developed. In particular, in [6], a butterfly-compressed representation of the CFIE was proposed and implemented in a randomized fast direct solver. An alternative approach is proposed in the present work. The Multilevel Nonuniform Grid (MLNG) approach has been successfully used in fast iterative solvers for acoustic and EM scattering problems [7–9]. Recently, it was found that the MLNG approach can be used for direct inversion of various integral operators. In particular, it turned out that the MLNG representation of the electric field integral equation (EFIE) operator is suitable for efficient direct inversion, since the inversion of MLNG-compressed integral operators reduces to a repeated inversion of banded matrices.

2. The MLNG Algorithm

Originally, the MLNG algorithm was developed to efficiently calculate integrals of the form [7–9]

$$\Psi_S(\mathbf{r}) = \int_S G(\mathbf{r}-\mathbf{r}')\varphi(\mathbf{r}')ds', \mathbf{r} \in S \quad (1),$$

where $G(\mathbf{r}-\mathbf{r}')$ is the free space scalar Green's function, and $\varphi(\mathbf{r}')$ is a scalar density. Instead of $G(\mathbf{r}-\mathbf{r}')$, we can use $D_{\mathbf{r},\mathbf{r}'}G(\mathbf{r}-\mathbf{r}')$, where $D_{\mathbf{r},\mathbf{r}'}$ is some differential operator of order 0 to 2.

The domain of integration S is divided into a hierarchy of progressively smaller subdomains $S_n^{(l)}$ of progressively lower levels, where $l = 0,1,\ldots,L$ is the level and n is the subdomain's index within the lth level. At the top level, $S_1^{(0)} = S$. Parent–child relations between subdomains are established. Each domain can have up to 8 child subdomains, which results in the formation of an octree. For each subdomain, except for l = 0 and 1, the near, far, and interpolation zones are defined, and, to each subdomain, a spherical interpolation grid $\Gamma_n^{(l)}$ is assigned. For subdomain $S_n^{(l)}$ with $l > 2$, the interpolation zone is the intersection of the near zone of the parent domain $S_{P(n)}^{(l-1)}$ with the far zone of $S_n^{(l)}$. Each grid $\Gamma_n^{(l)}$ allows interpolation to the observation points on S within the interpolation zone of $S_n^{(l)}$ and to the grids $\Gamma_m^{(l-1)}$ assigned to the parent domains $S_m^{(l-1)}$. We denote by $I_l(\mathbf{r})$ the set of the indices of lth-level domains to whose interpolation zone the point $\mathbf{r} \in S$ belongs. We form the union of nonintersecting sets $\Sigma(\mathbf{r}) = \bigcup_{l=2}^{L} \bigcup_{n \in I_l(\mathbf{r})} S_n^{(l)}$ .We also define the set $D(\mathbf{r}) = S \setminus \Sigma(\mathbf{r})$. Thus, integral (1) at a point $\mathbf{r} \in S$ can be represented by the sum

$$\Psi_S(\mathbf{r}) = \Psi_{D(\mathbf{r})}(\mathbf{r}) + \sum_{l=2}^{L} \sum_{n \in I_l(\mathbf{r})} \Psi_{S_n^{(l)}}(\mathbf{r}). \quad (2)$$

Note that the subscript of the first term on the right-hand side of (2)—the set of integration—depends on $\mathbf{r}$, as well as the selected set of domains $S_n^{(l)}$.

When calculating (2), only the first term is calculated by the direct integration; all other terms are evaluated by the interpolation to **r** of the amplitude-and-phase compensated field values on the grids $\Gamma_n^{(l)}$. The result of interpolation is multiplied by the phase-and-amplitude restoration factor, which depends on the point **r** and the subdomain $S_n^{(l)}$.

3. MLNG Representation and Direct Inversion of the Impedance Operator

We consider the problem of EM scattering by a perfectly conducting open surface or a set of such surfaces, *S*, in free space, formulated in the form of an EFIE. The impedance operator Z, which relates the surface current **J** and the tangential electric component of the incident field on *S*, fully complies with representation (1).

In accordance with the MLNG algorithm and representation (2), operator Z at a surface point $\mathbf{r} \in S$ is calculated in the form

$$(\mathrm{Z}\mathbf{J})(\mathbf{r}) = (\tilde{\mathrm{Z}}\mathbf{J})(\mathbf{r}) + \sum_{l=2}^{L} \sum_{n \in I_l(\mathbf{r})} U_n^{(l)}(\mathbf{r}) V_n^{(l)} \mathbf{J}. \quad (3)$$

Here, $\tilde{\mathrm{Z}}$ is the "diagonal" operator, which determines the tangential electric field produced at the point **r** by the currents on $D(\mathbf{r})$. The area of $D(\mathbf{r})$ is on the order of the area of a bottom-level domain. After discretization, the diagonal operator $\tilde{\mathrm{Z}}$ is represented by a banded matrix. The operator $V_n^{(l)}$ maps the current in the domain $S_n^{(l)}$ to values of the vector potential **A** and scalar potential $\mathbf{\Phi}$ (being part of the impedance operator) on the grid $\Gamma_n^{(l)}$. The interpolation-and-projection operator $U_n^{(l)}$ maps the values of the potentials at the grid $\Gamma_n^{(l)}$ to surface currents within the interpolation zone of $S_n^{(l)}$.

The key property of representation (3) is that the sum of all operators $U_n^{(l)} V_n^{(l)}$ of the lth level after discretization is represented by a banded matrix. The band width is determined by the maximum number of grid points in $\Gamma_n^{(l)}$ at a fixed level l.

The inversion of the impedance operator in the MLNG-compressed representation (3) begins with the inversion of the diagonal operator $\tilde{\mathrm{Z}}$, which requires inverting the corresponding banded matrix. At the successive steps, corresponding to levels from $l = L$ to 2, we invert operators approximated by banded matrices. The algorithm efficiency is entirely determined by the number of grid points participating in the interpolation and especially by the rate of its growth from level to level. It determines the increase in the band widths of the banded matrices. A gain in efficiency over the traditional methods based of the Gaussian elimination is achieved each time when the number of grid points increases slower than the square of the domain's diameter. Except for some special cases, this requirement can be achieved by an optimal choice of interpolation grids. In particular, if the number of grid points increases proportionally to the domain's diameter, the computational complexity is estimated as $O(N^{1.5} \log N)$.

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