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Spaced-Position Coherence Function of the Transionospheric Signal under Ionospheric Fluctuations with Finite Correlation Radius

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Abstract

An analytic technique recently developed for solving the nonlocal Markov equation for the mean field is further extended for treating the equation for the spaced position coherence function of the random field. The model rigorous explicit solution is presented for this function, which takes account of the finite values of the longitudinal correlation radius of fluctuations. It is also discussed how this theory can be employed for further modernization of the earlier developed simulators of the stochastic signals of very high frequencies, propagating through fluctuating plasma.

1. Introduction

In the classic books on wave propagation in random media [1, 2], the diffusive (delta-correlated) Markov approximation for the field statistical moments is discussed. This consideration is confined by the case of the homogeneous background medium and delta-correlated fluctuations. As far as the case of finite correlation radius of fluctuations is concerned, it is solely the appropriate integro-differential equation for the mean field that can be found there in the cited above books. Along with this, no explicit solution to such an equation is available.

Recently the nonlocal equation for the mean field was rigorously formulated and solved for the case of homogeneous background medium in [3], and its asymptotic solution was also presented in [4] for the case of smoothly inhomogeneous background medium (ionosphere).

To extend this activity, here below the rigorous solution to the first second-order spaced position coherence function will be presented for the case of finite values of the longitudinal correlation radius of fluctuations. The consideration here is confined by the case of the homogeneous background medium. However, in principle, it can be done in the same style, as this was done for the mean field for the case of a smoothly inhomogeneous background medium [4]. This is planned to also be done later.

2. Integro-Differential Equation for the Spaced Position Coherence Function

In this section, the second-order statistical moment of the wave field propagated in a stochastic medium is subject to consideration. Assuming a realistic case of VHF wave propagation in the homogeneous plasma layer, disturbed by fluctuations of its dielectric permittivity, the nonlocal Markov equation for the first second-order spaced-position coherence function is derived, and its analysis is done. Like in the problem of the mean field [3, 4], the appropriate nonlocal equation for the coherence function is derived and solved. The procedure of its derivation is similar to that for the mean field in the main course, although it differs in some substantial details.

To construct a fully rigorous (not asymptotic) solution to the appropriate integro-differential equation, here it is considered the case, where the background medium is a homogeneous isotropic cold plasma with the given constant electron density, which also contains the zero-mean statistically homogeneous stochastic component of the electron density.

2.1 Derivation of the Stochastic Parabolic Equation

The initial stochastic equation for the consideration below is as follows:

$$\Delta E + k^2[\varepsilon_0 + \varepsilon(\mathbf{r})]E = 0\,. \quad (1)$$

Here $k = \frac{\omega}{c}$, ω is the circle frequency, c is the light velocity in a vacuum, and $\varepsilon_0 = \text{const}$.

The background cold plasma is characterized by the constant dielectric permittivity of the form

$$\varepsilon_0 = 1 - \frac{4\pi q_e^2}{m_e \omega^2} N_0, \quad (2)$$

and the stochastic component of the dielectric permittivity is presented in the form

$$\varepsilon(\mathbf{r}) = (\varepsilon_0 - 1)\widetilde{\Delta N(\mathbf{r})}, \quad (3)$$

where $\widetilde{\Delta N(\mathbf{r})} = \frac{\Delta N(\mathbf{r})}{N_0}$. Fluctuations of the fractional electron density $\widetilde{\Delta N(\mathbf{r})}$ are assumed to be statistically homogeneous, i.e. their correlation function $\langle \widetilde{\Delta N(\mathbf{r})} \widetilde{\Delta N(\mathbf{r}')} \rangle$ is solely the function of the difference variable $\mathbf{r} - \mathbf{r}'$.

In the following consideration, the approximation of the parabolic equation is applied to equation (1). To obtain this, it is also assumed that all the local random inhomogeneities have spatial scales much larger, than the incident field wavelength. Then, introducing the substitution

$$E(\mathbf{r}) = E_0(\mathbf{r}) v(\mathbf{r}), \tag{4}$$

where E_0 represents the solution to the deterministic equation for the background medium

$$\Delta E_0 + k^2 \varepsilon_0 E_0 = 0, \tag{5}$$

which is, for simplicity, accepted to be the plane wave, propagating along the z-axis of the Cartesian coordinates $\mathbf{r} = (x, y, z)$ as follows:

$$E_0(\mathbf{r}) \cong U_0 \exp\left(ik\sqrt{\varepsilon_0} z\right). \tag{6}$$

The transversal to the line of sight variables are $\boldsymbol{\rho} = (x, y)$.

Taking all the said into account, the principal approximation to the solution of (1) takes the form

$$2ik\sqrt{\varepsilon_0}\frac{\partial v}{\partial z} + \Delta_{\perp} v + k^2 \varepsilon(\mathbf{r}) v = 0. \tag{7}$$

This stochastic equation is further employed in deriving the parabolic equation for the spaced position coherence function.

The stochastic parabolic equation (7) can be re-written for the pair of functions $v_1(\boldsymbol{\rho}_1, z)$ and $v_2^*(\boldsymbol{\rho}_2, z)$:

$$2ik\sqrt{\varepsilon_0}\frac{\partial v_1(\boldsymbol{\rho}_1, z)}{\partial z} + \Delta_{\boldsymbol{\rho}_1} v_1(\boldsymbol{\rho}_1, z) + k^2 \varepsilon(\boldsymbol{\rho}_1, z) v_1(\boldsymbol{\rho}_1, z) = 0\,, \tag{8}$$

$$2ik\sqrt{\varepsilon_0}\frac{\partial v_2^*(\boldsymbol{\rho}_2, z)}{\partial z} - \Delta_{\boldsymbol{\rho}_2} v_2^*(\boldsymbol{\rho}_2, z) - k^2 \varepsilon(\boldsymbol{\rho}_2, z) v_2^*(\boldsymbol{\rho}_2, z) = 0\,, \tag{9}$$

each of them depends on its own transverse (vector) variable, and the asterisk means the complex conjugate. Insofar as the dispersive medium is considered, all the quantities in (8) and (9) depend on the frequency ω, and $(\boldsymbol{\rho}, z) = \mathbf{r}$.

2.2 Derivation of the Nonlocal Equation for the Coherence Function

The procedure of obtaining the closed-form pure (diffusive, or local), or integro-differential (nonlocal) momenta equations is known. For the case of the homogeneous background medium, it may be found in [1, 2]. This procedure is employed here to derive the nonlocal equation for the first second-order transversal spaced position coherence function. It is of use to also point out that this coherence function structure is the 3D body for the statistically homogeneous fluctuations in the plane orthogonal to the line of sight, whereas it becomes the 5D structure, if there is no statistical homogeneity in the plane orthogonal to the line of sight.

To introduce this function, first, the quantity $w(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z)$ is defined as follows:

$$w(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = v(\boldsymbol{\rho}_1, z) v^*(\boldsymbol{\rho}_2, z). \tag{10}$$

Employing this, the spaced position transverse coherence function is obtained as an ensemble average of (10):

$$\Gamma(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \langle w(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) \rangle. \tag{11}$$

To derive the closed-form equation for this two-position coherence function, the equations (8) and (9) should be used in a known way [1, 2]. Then for the following consideration, the random field is introduced as follows

$$h(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = (\varepsilon_0 - 1)\, [\widetilde{\Delta N(\boldsymbol{\rho}_1}, z) - \widetilde{\Delta N(\boldsymbol{\rho}_2}, z)] \tag{12}$$

and the correlation function for this random field is determined of the form

$$\psi_h(\boldsymbol{\rho}_1, \boldsymbol{\rho}_1', \boldsymbol{\rho}_2, \boldsymbol{\rho}_2', z, z') = \langle h(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) h(\boldsymbol{\rho}_1', \boldsymbol{\rho}_2', z') \rangle. \tag{13}$$

From the structure of the representation (12) in the case of fully statistically homogeneous fluctuations $\widetilde{\Delta N(\boldsymbol{\rho}}, z)$ in (12) it follows that the correlation function ψ_h in (13) solely depends on the difference variables $\boldsymbol{\rho}_1 - \boldsymbol{\rho}_1'$, $\boldsymbol{\rho}_2 - \boldsymbol{\rho}_2'$, $\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2$ and $z - z'$.

Insofar as in further considerations the correlation function (13) would be taken at specific values of its arguments, it is straightforward to define the following functions:

$$\varphi_h(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2, z - z') = \psi_h(\boldsymbol{\rho}_1, \boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}_2, z - z')$$

$$\Phi_h(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2) = \int_{-\infty}^{+\infty} \varphi_h(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2, z - z') dz' \tag{14}$$

The last function $\Phi_h(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)$ is actually an effective transversal structure function of fluctuations of the fractional electron density. The explicit formulae for the functions (14) can be found from the appropriate relationships above.

Furthermore, the stochastic equation for the function $w(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z)$, defined in (10), is written as follows:

$$2ik\sqrt{\varepsilon_0}\frac{\partial w}{\partial z}+(\Delta_{\rho_1}-\Delta_{\rho_2})w+k^2hw=0. \qquad (15)$$

Averaging equation (15) results in the new unclosed form equation

$$2ik\sqrt{\varepsilon_0}\,\frac{\partial\langle w\rangle}{\partial z}+(\Delta_{\rho_1}-\Delta_{\rho_2})\langle w\rangle+k^2\langle hw\rangle=0 \qquad (16)$$

with the arguments omitted but readily restored from the previous designations.

The procedure of treating the last term here, which employs the Furutsu-Novikov relationship in order to split the cross-correlation in the last term here, is known [1, 2]. Applying this procedure here not on the same step as employed when deriving the diffusive Markov momenta equations, but on the next step, the closed-form integro-differential (nonlocal) equation for the spaced-position second-order coherence function (11) is obtained as follows:

$$\frac{\partial\Gamma}{\partial z}+\frac{1}{2ik\sqrt{\varepsilon_0}}(\Delta_{\rho_1}-\Delta_{\rho_2})\Gamma+\frac{1}{4}\frac{k^2}{\varepsilon_0}\int_0^z\varphi_h(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2,z-z')\cdot$$
$$\Gamma(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,z')\exp\left\{-\frac{k^2}{\varepsilon_0}\frac{z-z'}{8}\Phi_h(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)\right\}dz'. \qquad (17)$$

This equation is reduced to the classic [1, 2] diffusive Markov equation for the second-order transversal spaced position coherence function, if it is accepted that the function $\varphi_h(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2,z-z')$ has the δ-function dependence on its longitudinal difference variable $z-z'$.

Additionally, it should also be mentioned that equation (17) has a structure, which is fairly similar to the appropriate nonlocal equation for the mean field shown in [2] (at the bottom of page 278, without number).

2.3 Solution to Nonlocal Equation (17)

To solve equation (17), reasonable simplifications are required to construct the explicit rigorous solution. To clearly perform necessary transformations, the initial condition for equation (17) is adopted in the form $\Gamma(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,0)=1$, which corresponds to having the incident field in the form of a plane wave, which propagates in the direction *z*.
Such a condition permits reducing this equation to a simpler form as follows:

$$\frac{\partial\Gamma(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2,s)}{\partial z}+\frac{1}{4}\frac{k^2}{\varepsilon_0}\int_0^z\varphi_h(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2,z-z')\cdot$$
$$\Gamma(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2,z')\exp\left\{-\frac{k^2}{\varepsilon_0}\frac{z-z'}{8}\Phi_h(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)\right\}dz'. \qquad (18)$$

The solution to equation (18) can be readily obtained for the case of the exponential model of the correlation function

$$\varphi_h(\boldsymbol{\rho},z-z')=\frac{\Phi_h(\boldsymbol{\rho})}{2l_{||}}e^{-\frac{|z-z'|}{l_{||}}}, \qquad (19)$$

with $\boldsymbol{\rho}=\boldsymbol{\rho}_1-\boldsymbol{\rho}_2$. The coefficient before the exponent is chosen in order to satisfy the normalization condition (14). Omitting all intermediate calculations, the formula

$$\Gamma(\boldsymbol{\rho},z)=\frac{1}{1-\frac{k^2\Phi_h(\boldsymbol{\rho})}{\varepsilon_0\ \ 8}l_{||}}e^{-\frac{1k^2}{8\varepsilon_0}\Phi_h(\boldsymbol{\rho})z}-\frac{\frac{k^2\Phi_h(\boldsymbol{\rho})}{\varepsilon_0\ \ 8}l_{||}}{1-\frac{k^2\Phi_h(\boldsymbol{\rho})}{\varepsilon_0\ \ 8}l_{||}}e^{-\frac{z}{l_{||}}} \qquad (20)$$

is obtained for the coherence function Γ.

This result presents the rigorous solution to the nonlocal equation (18) for the transversal spaced position coherence function, which explicitly shows the dependence of this function on the finite values of the correlation function on the longitudinal correlation radius $l_{||}$ of the fluctuations of the electronic density. At $l_{||}$=0 the second term in (20) vanishes, whereas the first one transforms into the solution to this coherence function, which corresponds to the classic solution for the case of delta-correlated in the direction *z* fluctuations [1]. In a similar style, the rigorous explicit solution can be constructed to the Gaussian model of the correlation function with the finite longitudinal correlation radius of fluctuations.

Unlike these two models, the case of the inverse power law spectrum of fluctuations does not permit constructing a rigorous solution to this function but requires the direct numerical solution to equation (18). The appropriate numerical codes for such calculations are available.

3. Implementation of the Solution into the Simulator of Random Signals

As was mentioned earlier in Abstract, in papers [5, 6] the simulator of VHF stochastic signals on the transionospheric paths of propagation has been presented, based on solving the appropriate Markov diffusive (local) equations for the wave field statistical moments and accounting, however, for the inhomogeneous background ionosphere.
The results of the present work form the physical background for building the extended version of such a simulator, based on the solutions to the appropriate nonlocal Markov momenta equations. The core point for its modification is incorporating the obtained here the analytical solution to the nonlocal (integro-differential) equation for the field spaced-position coherence function of a random field and earlier obtained solution to the nonlocal equation for the mean field [3, 4]. Based on these analytical results mentioned, the numerical codes for appropriate calculations will be built. This novel simulator would generate the field random time series more relevant to the physically adequate situation of the

electron density fluctuations, characterized by the finite values of the longitudinal correlation radius.

Here it should be additionally pointed out that, rigorously speaking, when generating the random time series of the field, propagating in the plasma layer with the stochastic component of the zero-mean electron density fluctuations, it is also required knowing the scintillation index S_4. Then the appropriate nonlocal solution to the fourth-order random field should be also constructed in order to also properly choose the value of S_4, and therefore, to properly specify the single point probability density function of the field fluctuations.

However, the procedure of specifying the type of the single point probability density function of the field fluctuations [5, 6] is itself quite empirical. Therefore, here it seems reasonable to meanwhile keep the procedure of finding the S_4 index the same as in just cited papers, i.e., obtaining this index from the solution to the equation for the fourth-order moment written in the diffusive (delta-correlated) Markov approximation.

4. Conclusion

The analytical technique developed in recent papers [3, 4] for calculating the mean field in a homogeneous background plasma layer, which also contains the zero-mean fluctuations of the electron density (dielectric permittivity), characterized by a finite value correlation radius, was further extended here in order to describe the spaced position coherence function of the random field. The nonlocal equation (17) for this function was derived and solved, which presents the next order approximation, as compared to the traditional δ- correlated (diffusive) Markov approximation. It takes account of the finite values of the longitudinal correlation radius of fluctuations of the electron density. As was mentioned above, the nonlocal solution (21) to equation (17) is the core quantity to further modify in proper way the simulator of VHF signals, propagating in the homogeneous layer of the background plasma with the zero mean fluctuations of the electron density (dielectric permittivity).

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