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A Macroscopic Field Equation in Spacetime Algebra

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Abstract

Geometric or Clifford algebra is a powerful, coordinate-free language for mathematical physics, offering more compact and insightful descriptions of natural laws than existing legacy formalisms in areas ranging from basic geometry and complex analysis to quantum mechanics and general relativity [1–5]. In its four-dimensional form employing the Minkowski metric, known as spacetime algebra, it is extraordinarily effective at describing electrodynamics in a relativistic framework, compressing Maxwell's equations into one simple expression that applies independent of any preferred inertial reference frame.

However, as the practitioners of spacetime algebra have typically come from either a pure mathematics or theoretical physics background, only the microscopic form of Maxwell's equation, that which applies to fundamental particles in a vacuum, has received widespread attention. The conventional macroscopic simplification used by engineers, wherein the effects of bound charges and currents in a dielectric medium are subsumed into phenomenological constituent parameters—the relative permittivity and permeability—is frequently neglected.

In this paper I describe a macroscopic form of Maxwell's equation in spacetime algebra where only free charges and currents appear explicitly.

1 Introduction

A growing number of researchers have concluded that geometric algebra has tremendous advantages over other common mathematical formalisms including, for the present purposes, the vector calculus developed by J. Willard Gibbs and Oliver Heaviside which is taught almost universally to all students of electrical engineering. The author shares their conviction that it can and should replace the traditional Gibbs-Heaviside vector form as the preferred language for electromagnetic discourse [6–8].

A comprehensive treatment of geometric algebra is beyond the scope of this paper. However, as it is likely to be unfamiliar to many in the intended audience, a brief summary of the most basic concepts needed for the purposes of this paper is given in Section 2. The reader is referred to the references for a more thorough description of its principles, theorems, and advantages over other algebraic systems [1–9].

2 Mathematical Background

The core element of spacetime algebra is the four-vector, a four-dimensional analog to the three-dimensional vectors commonly employed in Gibbs-Heaviside vector calculus. Spacetime four-vectors may be expanded in any given inertial reference frame in terms of the basis vectors $\{\gamma_\mu\}$,

$$a = a^\mu \gamma_\mu = a^0\gamma_0 + a^1\gamma_1 + a^2\gamma_2 + a^3\gamma_3 \tag{1}$$

where the superscripts represent *contravariant* indices (as in tensor analysis) rather than exponentiation, and the Einstein summation convention has been applied. For simplicity, this vector may also be written as the quadruplet of contravariant coefficients $\left(a^0, a^1, a^2, a^3\right)$. Inner products obey the Minkowski metric with signature $(+---)$,

$$\gamma_\mu \cdot \gamma_\mu = \gamma_u^2 = \begin{cases} +1 & \mu = 0 \\ -1 & \mu \neq 0 \end{cases} \tag{2}$$

where γ_0 is the temporal component. The square magnitude of any spacetime coordinate vector r generates the *proper time*, $\tau^2 = (ct)^2 - x^2 - y^2 - z^2$, which is invariant in all inertial reference frames.

The outer, or *wedge* product $a \wedge b$ generates a *bivector*, an oriented surface in the plane defined by the vectors a and b, directed rotationally from a toward b, and with magnitude equal to the area of the parallelogram having a and b as its first two edges. Such products are associative, distributive, and anti-commutative. Bivectors are called grade 2, whereas vectors are grade 1, in view of their dimensionality. Spacetime algebra supports objects up to grade 4 (the wedge product of four independent vectors).

Upright, boldface characters represent *simple bivectors* [8]. Most important are those that define the spatiotemporal planes,

$$\sigma_1 = \gamma_1 \wedge \gamma_0 = \mathbf{x} \tag{3a}$$

$$\sigma_2 = \gamma_2 \wedge \gamma_0 = \mathbf{y} \tag{3b}$$

$$\sigma_3 = \gamma_3 \wedge \gamma_0 = \mathbf{z} \tag{3c}$$

As noted above, the principle bivectors σ_k equate to the coordinate axes ($\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$) in three-dimensional Euclidean

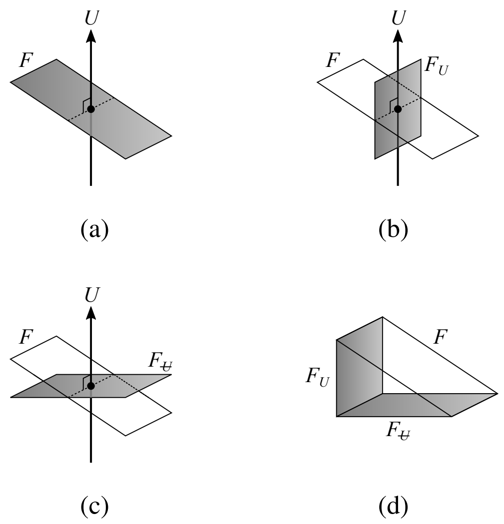

Figure 1. Illustration of projection and rejection operations. (a) A bivector plane F and vector U. (b) Projection of F onto U. (c) Rejection of F from U. (d) The original bivector F is the sum of its projection and rejection with respect to U.

space (as opposed to four-dimensional Minkowski space). Note that all such bivectors have positive square norm, $\sigma_k^2 = 1$. It is important to remember that any decomposition of a spacetime quantity into Euclidean components written in boldface characters requires first the selection of a temporal axis, or *worldine*, γ_0, and is thus specific to the inertial reference frame of a particular observer. In contrast, spacetime four-vectors are frame-independent.

The *geometric product* or *Clifford product* for four-vectors is the sum of the inner and outer products,

$$ab = a \cdot b + a \wedge b \tag{4}$$

resulting in a scalar-bivector composite object called a *multivector*. The geometric products of orthogonal basis vectors are often written with multiple subscripts for simplicity, e.g. $\gamma_1\gamma_2\gamma_3 = \gamma_{123}$.

The geometric product is associative and distributive (but not usually commutative). This allows us to generalize the geometric product for higher-order multivectors. The product of a grade-k object with a grade-l object yields terms with grades ranging from $k-l$ to $k+l$. The dot and wedge product notation is reserved for the lowest and highest-order of these terms, respectively [6, 8].

The *projection* of F onto U may be written with a subscript notation [8],

$$F_U = (F \cdot U)\,U^{-1} \tag{5}$$

where $U^{-1} = U/U^2 = U/(U \cdot U)$. See Figure 1, where F and U are visually interpreted as a bivector plane segment and vector, respectively. The *rejection* of F from U may be written with slash-subscript notation [8],

$$F_{\not U} = (F \wedge U)\,U^{-1} \tag{6}$$

Any multivector may be written as the sum of its projection and rejection with a respect to a given vector, $F = F_U + F_{\not U}$.

The spacetime differential operator is given by

$$\square = \gamma_\mu \partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right) \tag{7}$$

Finally, the unique (up to a sign) unit four-volume in spacetime, γ_{0123}, shall be called the unit *pseudoscalar* and given the label i (noting that $i^2 = -1$).

3 Maxwell's Equation in a Vacuum

The microscopic/vacuum form of Maxwell's equation in SI units is

$$\square F = \mu_0 J \tag{8}$$

where $F = \frac{1}{c}\mathbf{E} + i\mathbf{B}$ is the bivector *Faraday field*, and $J = (c\rho, J_x, J_y, J_z)$ is the four-current density, comprising the charge density scaled by c as the temporal component, and the traditional current density as the spatial components. The left-hand side of (8) is the spacetime gradient of F, having the form of a geometric product, which in turn expands into inner and outer products, $\square F = \square \cdot F + \square \wedge F$, called the spacetime divergence and curl, respectively. These terms may then be separated by grade, producing the *inhomogeneous* Maxwell's equation ($\square \cdot F = \mu_0 J$) and *homogeneous* Maxwell's equation ($\square \wedge F = 0$).

Remember that the decomposition of F into $\mathbf{E}$ and $\mathbf{B}$ components depends on the chosen worldline. The Faraday field exists as a two-dimensional surface in spacetime; the electric and magnetic fields measured by a particular observer are merely the components of that surface which are parallel and perpendicular to the observer's worldline. That is, $F_{\gamma_0} = \frac{1}{c}\mathbf{E}$ and $F_{\not\gamma_0} = i\mathbf{B}$.

4 Maxwell's Equation in Media

The four-current density, J, includes both free and bound sources. To rewrite (8) in a form that only depends on free sources, we must derive a constitutive relation that reproduces the known response of the material in its rest frame, but in a form that can be transformed according to the Lorentz boost equations. Classically, the auxiliary fields are related to the intrinsic electric and magnetic fields by the following equations,

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_0(1+\chi_e)\,\mathbf{E} = \varepsilon_0\varepsilon_r\mathbf{E} = \varepsilon\mathbf{E} \tag{9a}$$

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1+\chi_m)\,\mathbf{H} = \mu_0\mu_r\mathbf{H} = \mu\mathbf{H} \tag{9b}$$

where $\mathbf{P}$ is the dielectric polarization density and $\mathbf{M}$ is the magnetization, representative of the electric displacement and magnetic field associated with bound sources. The parameters χ_e and χ_m are the electric and magnetic susceptibilities, respectively. We have assumed for the purpose of this paper that the material is linear and isotropic, so that ε and μ are scalars.

Let us define $\mathscr{D} = c\mathbf{D} + i\mathbf{H}$ as the *electromagnetic displacement bivector* and $\mathscr{M} = -\mathbf{cP} + i\mathbf{M}$ as the *magnetization-polarization bivector*. Adding them together and substituting from (9), we have

$$\mu_0 (\mathscr{D} + \mathscr{M}) = \mu_0 c (\mathbf{D} - \mathbf{P}) + i\mu_0 (\mathbf{H} + \mathbf{M}) \tag{10}$$
$$= \tfrac{1}{c}\mathbf{E} + i\mathbf{B} = F \tag{11}$$

Substituting this into the inhomogeneous Maxwell's equation, we have

$$\Box \cdot (\mathscr{D} + \mathscr{M}) = J = J^{free} + J^{bound} \tag{12}$$

or, since the elements of $\mathscr{M}$ account for the bound sources while $\mathscr{D}$ depends only on the free sources,

$$\Box \cdot \mathscr{D} = J^{free} \tag{13a}$$

$$\Box \cdot \mathscr{M} = J^{bound} \tag{13b}$$

The homogeneous Maxwell's equation ($\Box \wedge F = 0$), having no dependence on sources at all, whether bound or free, remains unchanged.

The constitutive relations (9) may be written in a unified form by substituting in the projection/rejection of the Faraday field in place of $\mathbf{E}$ and $\mathbf{B}$,

$$\mathscr{D} = c\mathbf{D} + i\mathbf{H} = c\varepsilon\mathbf{E} + \tfrac{i}{\mu}\mathbf{B} = \tfrac{1}{\mu}\left(n^2 F_{\gamma_0} + F_{\not\gamma_0}\right) \tag{14}$$

where $n = c\sqrt{\mu\varepsilon}$ is the material's *proper index of refraction* (the index of refraction that applies in its rest frame).

So far, everything we have done in this section has been in the rest frame of the material. When boosted into another reference frame, the worldine of the material transforms into the normalized four-velocity of the boost, $\gamma_0 \to -\frac{1}{c}U$. Therefore, we have

$$\mathscr{D} = \tfrac{1}{\mu}\left(n^2 F_U + F_{\not U}\right) \tag{15}$$

For maximum utility, it is desirable to unify the homogeneous and inhomogeneous equations ($\Box \cdot \mathscr{D} = J^{free}$ and $\Box \wedge F = 0$) into one. To that end, let us now define the *macroscopic Faraday field* [8],

$$\mathscr{F} = \eta\left(n^{-1}\mathscr{D}_U + \mathscr{D}_{\not U}\right) = c\left(F_U + n^{-1}F_{\not U}\right) \tag{16}$$

where $\eta = \sqrt{\mu/\varepsilon}$ is the *proper wave impedance of the medium*. In the rest frame, $\mathscr{F} = \mathbf{E} + i\eta\mathbf{H}$. We also define a *macroscopic differential operator* [8],

$$\blacksquare = n\Box_U + \Box_{\not U} \tag{17}$$

The filled-square notation is intended to remind us that it applies to the fields in a medium as opposed to empty space. The directional derivatives $\Box_U$ and $\Box_{\not U}$ are found by the same projection and rejection formulas as ordinary multivectors, e.g. (5) and (6).

We now compute the spacetime divergence of the macroscopic Faraday field,

$$\blacksquare \cdot \mathscr{F} = \eta\left(n\Box_U + \Box_{\not U}\right) \cdot \left(\tfrac{1}{n}\mathscr{D}_U + \mathscr{D}_{\not U}\right) \tag{18a}$$
$$= \eta\left(\Box_U \cdot \mathscr{D}_U + n\Box_U \cdot \mathscr{D}_{\not U} + \tfrac{1}{n}\Box_{\not U} \cdot \mathscr{D}_U + \Box_{\not U} \cdot \mathscr{D}_{\not U}\right) \tag{18b}$$
$$= \eta\left(\Box_U \cdot \mathscr{D}_U + n\Box_U \cdot \mathscr{D}_{\not U} + \tfrac{1}{n}\Box_{\not U} \cdot \mathscr{D}_U + \Box_{\not U} \cdot (\mathscr{D} - \mathscr{D}_U)\right) \tag{18c}$$
$$= \eta\left(\Box_U \cdot \mathscr{D}_U + n\Box_U \cdot \mathscr{D}_{\not U} + \left(\tfrac{1}{n} - 1\right)\Box_{\not U} \cdot \mathscr{D}_U + \Box_{\not U} \cdot \mathscr{D}\right) \tag{18d}$$

This expression may be simplified through some identities related to projection and rejection [8],

$$\Box_U \cdot \mathscr{D}_U = \Box_U \cdot \mathscr{D} \tag{19a}$$

$$\Box_U \cdot \mathscr{D}_{\not U} = 0 \tag{19b}$$

$$\Box_{\not U} \cdot \mathscr{D}_U = (\Box \cdot \mathscr{D})_U \tag{19c}$$

Application of these identities yields the following simplification,

$$\blacksquare \cdot \mathscr{F} = \eta\left(\Box_U \cdot \mathscr{D} + \left(n^{-1} - 1\right)(\Box \cdot \mathscr{D})_U + \Box_{\not U} \cdot \mathscr{D}\right) \tag{20a}$$
$$= \eta\left(\Box \cdot \mathscr{D} + \left(n^{-1} - 1\right)(\Box \cdot \mathscr{D})_U\right) = \eta\left(J^{free} + \left(n^{-1} - 1\right)J_U^{free}\right) \tag{20b}$$
$$= \eta\left(n^{-1}J_U^{free} + J_{\not U}^{free}\right) = \mathscr{J} \tag{20c}$$

The final expression for $\mathscr{J}$ is the macroscopic free source vector. Typically, in most engineering problems, we will be considering source-free regions (excluding bound sources) for which $\mathscr{J} = 0$.

Now we compute the spacetime curl of $\mathscr{F}$,

$$\blacksquare \wedge \mathscr{F} = \tfrac{c}{n}\left(n\Box_U + \Box_{\not U}\right) \wedge \left(nF_U + F_{\not U}\right) \tag{21a}$$
$$= \tfrac{c}{n}\left(n^2\Box_U \wedge F_U + n\Box_U \wedge F_{\not U} + n\Box_{\not U} \wedge F_U + \Box_{\not U} \wedge F_{\not U}\right) \tag{21b}$$
$$= \tfrac{c}{n}\left(n^2\Box_U \wedge F_U + n\Box_U \wedge F_{\not U} + n\Box_{\not U} \wedge (F - F_{\not U}) + \Box_{\not U} \wedge F_{\not U}\right) \tag{21c}$$
$$= \tfrac{c}{n}\left(n^2\Box_U \wedge F_U + n\Box_U \wedge F_{\not U} + n\Box_{\not U} \wedge F + (1 - n)\Box_{\not U} \wedge F_{\not U}\right) \tag{21d}$$

Once again, this expression may be simplified with some identities [8],

$$\Box_U \wedge F_U = 0 \tag{22a}$$

$$\Box_U \wedge F_{\not U} = \Box_U \wedge F \tag{22b}$$

$$\Box_{\not U} \wedge F_{\not U} = (\Box \wedge F)_{\not U} \tag{22c}$$

Thus,

$$\blacksquare \wedge \mathscr{F} = \tfrac{c}{n}\left(n\Box_U \wedge F + n\Box_{\not U} \wedge F + (1 - n)(\Box \wedge F)_{\not U}\right) \tag{23a}$$
$$= \tfrac{c}{n}\left(n\Box \wedge F + (1 - n)(\Box \wedge F)_{\not U}\right) = 0 \tag{23b}$$

Combining the divergence (20) and curl (23) into a spacetime gradient ($\blacksquare \cdot \mathscr{F} + \blacksquare \wedge \mathscr{F} = \blacksquare\mathscr{F}$), we have the following unified form of Maxwell's equation in dielectric media [8],

$$\blacksquare\mathscr{F} = \mathscr{J} \tag{24}$$

It is also possible, since the curl of $\mathscr{F}$ is zero (23), to define a macroscopic four-vector potential, $\mathscr{A}$, such that $\mathscr{F} = \blacksquare\mathscr{A}$. Under the Lorenz gauge condition ($\blacksquare \cdot \mathscr{A} = 0$), we have $\blacksquare^2\mathscr{A} = \mathscr{J}$.

5 Application to Plane Waves in Media

The utility of this simple form is exemplified by applying it to the basic problem of wave propagation in a dielectric. We begin by writing an ansatz of the Faraday field in polar form,

$$\mathscr{F} = \mathbf{f}e^{i\varphi} \tag{25}$$

In this expression, $\mathbf{f}$ is a simple bivector, $i = \gamma_{0123}$ is the unit pseudoscalar (the unit-magnitude four-volume in spacetime), and the scalar phase φ is known as the *complexion* of $\mathscr{F}$. To derive a plane-wave solution, we assume that the amplitude $\mathbf{f}$ is constant, so that all variations throughout space and time are contained in the complexion, φ. Applying Maxwell's equation in a source-free region ($\blacksquare\mathscr{F} = 0$), we have

$$\blacksquare\mathscr{F} = \blacksquare\left(\mathbf{f}e^{i\varphi}\right) = (\blacksquare\varphi)\, i\mathbf{f}e^{i\varphi} = 0 \tag{26a}$$

$$\therefore (\blacksquare\varphi)\,\mathbf{f} = 0 \tag{26b}$$

Neither $\mathbf{f}$ nor $\blacksquare\varphi$ can be zero in the above equation, for in the first case the field would be zero everywhere, and in the latter case the field would be static throughout space and time, both trivial solutions. Instead, we must conclude that both are null vectors (nonzero in value, but having zero norm), for otherwise we could invert one in the above equation and find that the other is zero.

We may thus write the bivector $\mathbf{f}$ as the product of two orthogonal vectors, one of which is null. Let $\mathbf{f} = s \wedge L = sL$ where L is the null vector. Then, to satisfy (26b), we must have $\blacksquare\varphi = L$. Integration yields

$$\varphi = K \cdot r + \varphi_0 \tag{27}$$

where r is the spacetime coordinate vector, φ_0 is a scalar constant of integration, and the dot product $K \cdot r$ (the grade-0 residual of the geometric product Kr) has been selected since φ is a scalar. K may be interpreted as the *four-wavevector* $(\omega/c, k_x, k_y, k_z)$ and is constrained by the derivative so that $\blacksquare\varphi = L$,

$$\blacksquare\varphi = \left(n\square_U + \square_{\underline{U}}\right)(K \cdot r) = nK_U + K_{\underline{U}} = L \tag{28}$$

The final expression for our plane wave solution, then, is

$$\mathscr{F} = sLe^{i(K\cdot r+\varphi_0)} \tag{29}$$

Let us illuminate this solution better by plugging in some practical values. We let $\varphi_0 = 0$ and assume that propagation is along the z-axis, $K = \frac{\omega}{c}\gamma_0 + k\gamma_3$. The complexion is thus

$$\varphi = K \cdot r = \omega t - kz \tag{30}$$

When the medium is at rest ($U = c\gamma_0$), we have

$$L = nK_U + K_{\underline{U}} = \frac{n\omega}{c}\gamma_0 + k\gamma_3 \tag{31}$$

Recall that L must be a null vector. Therefore, $L^2 = 0$ and $kc = n\omega$. Letting our reference polarization $s = \gamma_1/k$, we have $sL = \gamma_{10} + \gamma_{13} = \sigma_1 + i\sigma_2$. Plugging these values into (29), we obtain

$$\mathscr{F} = (\sigma_1 + i\sigma_2)\, e^{i(\omega t - kz)} \tag{32}$$

Or, to put it in more familiar terms,

$$\mathbf{E} = \mathbf{x}\cos(\omega t - kz) - \mathbf{y}\sin(\omega t - kz) \tag{33a}$$

$$\eta\mathbf{H} = \mathbf{x}\sin(\omega t - kz) + \mathbf{y}\cos(\omega t - kz) \tag{33b}$$

which are formulas for a circularly polarized plane wave.

6 Conclusion

As in the example of Section 5, in which the waves were propagating in a bulk medium, it is usually easiest to solve first for the fields in the rest frame of the material, then apply well-known Lorentz-boost formulae to find the fields when the medium is in motion. The true value of (24) for engineering is not so much that it applies in relativistic scenarios, but that it offers a much more compact form of Maxwell's equation for dielectric media that is easier to work with than the equivalent system of equations in Gibbs-Heaviside form. It also enables one to more easily solve problems with dynamic interfaces, in which one medium is moving with respect to another. The presentation will give an example where a wave incident from one medium (air, for example) refracts into another medium in motion against it (say, flowing water). The relativistic drag of the electromagnetic waves in the target medium alters the angle of refraction (albeit very slightly, in most practical cases) relative to the angle of refraction that would occur when both media are at rest in the same reference frame.

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