



On the optimality of IRS-user association for rank-1 channel conditions

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Abstract

This paper deals with the optimization of intelligent reflecting surfaces (IRSs) in a cellular communication scenario where only the signal reflected from the IRSs allows the connectivity between base station and users. It considers a particularly simple configuration, IRS-user association, where each IRS steers the reflected signal toward a given user. For the particular case of two users and two IRSs, optimality conditions for such configuration are found. Moreover, simulation results in realistic conditions show that IRS-user association is optimal with high probability.

1 Introduction

Beyond-5G cellular communications are foreseen to serve unprecedented throughput demand by tens of user equipments (UE) and internet-of-things (IoT) devices sharing the same physical resources. In order to accommodate so many UEs, high-frequency bands such as sub-THz and THz bands are expected to play an important role [1]. The availability of plenty of spectrum resources, however, comes at the price of challenging channel conditions: at (sub-)THz bands, scattering becomes weak and most of the received power either arrives through the LoS component, or through very few reflected rays. Moreover, when beamforming is employed, the signal power is concentrated in a specific direction, and the effect of multipath is further reduced making the channel matrices very low-rank.

In order to increase the coverage and favor connectivity between the base station (BS) and the UEs, a popular research field considers using intelligent reflecting surfaces (IRSs), which are surfaces composed by many elementary units made up of metamaterials, able to reflect the impinging wave towards a desired direction through electronic control. The contribution of IRSs as *smart mirrors* [2] is fundamental especially when the BS is not in LoS with the UE. IRSs can in principle be useful both outdoor and indoor.

A lot of papers concentrate on IRSs in a variety of settings [3]. Most papers deal with IRS optimization in a rich-scattering environment (i.e., at GHz bands), where the goal is typically to maximize the received signal-to-interference-plus-noise ratio (SINR), and the vector of phase shifts applied by each IRS element to the impinging signal is the

optimization parameter. No closed-form solution is available and iterative algorithms are usually employed to find a good solution. The same approach can be also applied at (sub-)THz bands, where, however, the different channel conditions suggest a heuristic solution in which, as already said, the individual IRS elements synergically cooperate to focus the reflected beam into a given direction.

In this paper, we consider the case in which i) there is no LoS between the BS and the UEs, and ii) the only received power at the UEs comes from reflection on the IRSs. While this hypothesis can be considered untrue for GHz bands, it becomes more and more close to reality for subTHz and THz bands, where contributions from reflections on walls and on other large, flat objects, typically come with significant attenuations. In particular, our case study considers the downlink of a communication system where a BS communicates with 2 UEs by exploiting 2 IRSs, each composed by a large number of elements. Assuming that the BS performs zero-forcing (ZF) precoding, we concentrate on the case in which each IRS is assigned to a given UE, and steers the reflected beam towards it, a scenario called IRS-user association in this paper. Our goal is to understand when IRS-user association is optimal, in the sense that it maximizes the received SINR, and when, instead, it is better for a given IRS to contribute to both users' received power. Simulation results in realistic conditions will show that IRS-user association is optimal with a relatively large probability.

2 Communication model

We consider the downlink of a wireless network in which a BS, equipped with a uniform linear array (ULA) of M isotropic antennas, transmits different data streams to two UEs. We suppose that the UEs are not in LoS with the BS, so that, to reach the UEs, the BS exploits two IRSs, each characterized by an $L \times L$ array of elements as shown in Fig. 1, aligned to have L horizontal rows. The signals propagating from the BS to the IRSs and from the IRSs to the UEs are assumed to undergo LoS conditions, free-space attenuation and, possibly, shadowing. For simplicity, we assume a 2D geometric model, without taking into account elevation. As a consequence, all L IRS rows see the same system. Generalizing to a 3D model is straightforward.

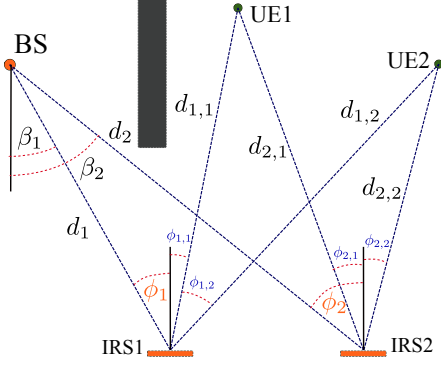


Figure 1. Network geometric model. The BS-UEs direct links are unavailable due to an obstacle represented by the black solid block. The dashed lines represent LoS paths.

Let the signal received at the UEs be:

$$\mathbf{y} = \mathbf{H}\mathbf{\Gamma}\mathbf{x} + \mathbf{n} \quad \text{where} \quad (1)$$

$\mathbf{x} = (x_1, x_2)^T$ is the 2×1 vector of transmitted symbols for the two UEs, which are assumed to be random complex Gaussian with zero-mean and unit variance; $\mathbf{y} = (y_1, y_2)^T$ is the 2×1 vector of received samples at the two UEs; \mathbf{H} is the $2 \times M$ channel matrix from the BS ULA to the users; $\mathbf{\Gamma}$ is the $M \times 2$ precoder and $\mathbf{n} = (n_1, n_2)^T$ is a vector of i.i.d. zero-mean Gaussian noise samples.

The precoder, $\mathbf{\Gamma}$ can be designed in many ways. We here consider the ZF precoder whose expression is given by

$$\mathbf{\Gamma} = \sqrt{\mathcal{P}} \frac{\mathbf{H}^{-1}}{\|\mathbf{H}^{-1}\|_F} \quad (2)$$

where \mathcal{P} is the transmitted power and $\|\cdot\|_F$ represents the Frobenius norm. While the expression for $\mathbf{\Gamma}$ in (2) is sub-optimal in terms of achievable rate, it has the advantage of completely removing inter-user interference at the receivers, and allowing for relatively simple analytical developments. Substituting (2) into (1), we obtain

$$\mathbf{y} = \frac{\sqrt{\mathcal{P}}}{\|\mathbf{H}^{-1}\|_F} \mathbf{x} + \mathbf{z} \quad (3)$$

Observe that the signal-to-noise ratio associated to (3) is the same for all users and is maximized by minimizing the value of $\|\mathbf{H}^{-1}\|_F$. The goal of this work is to investigate the minimization of $\|\mathbf{H}^{-1}\|_F$ by properly configuring the phase shift applied by the IRS elements to the impinging signal.

For future use, we define the norm-1 length- M column vector $\mathbf{s}(\delta, M, \beta)$, whose m -th entry, $m = 0, \dots, M-1$, is

$$[\mathbf{s}(\delta, M, \beta)]_m = \frac{1}{\sqrt{M}} e^{-j2\pi\delta(m-1)\sin\beta} \quad (4)$$

The k -th row of \mathbf{H} , $k = 1, 2$, corresponding to the k -th UE's channel, is given by [4]:

$$\sum_{n=1}^2 a_{k,n} \mathbf{u}_{k,n}^H \mathbf{\Theta}_n \mathbf{u}_n \mathbf{v}_n^H \quad \text{where} \quad (5)$$

- $a_{k,n}$ models the attenuation experienced by the signal reflected by IRS n and received by UE k , given by

$$a_{k,n} = \frac{\rho_{k,n} L^2 \Delta^2 \sqrt{M \cos \phi_n \cos \phi_{k,n}}}{4\pi d_n d_{k,n}} e^{-j\frac{2\pi}{\lambda}(d_n + d_{k,n})} \quad (6)$$

being λ the signal wavelength, Δ the IRS inter-element spacing, d_n the distance between the BS and the n -th IRS, $d_{k,n}$ the distance between the n -th IRS and the k -th UE, ϕ_n the angle of arrival of the signal at the n -th IRS, $\phi_{k,n}$ the angle of departure from the n -th IRS towards the k -th UE (see Fig. 1), $\rho_{k,n}$ a coefficient taking into account IRS efficiency and shadowing;

- $\mathbf{v}_n = \mathbf{s}(d/\lambda, M, \beta_n)$ represents the spatial signature of the signal transmitted from the BS, where d is the inter-antenna spacing in the BS ULA, and β_n is the angle of departure of the signal from the BS;
- $\mathbf{u}_n = \mathbf{s}(\Delta/\lambda, L, \phi_n)$ represents the spatial signature of the signal received at the n -th IRS;
- $\mathbf{\Theta}_n$ is the $L \times L$ diagonal matrix collecting the phase shifts introduced by the elements of any row of the n -th IRS (since all rows see the same system, we can assume they act in the same way on the impinging signal);
- $\mathbf{u}_{k,n} = \mathbf{s}(\Delta/\lambda, L, \phi_{k,n})$ represents the spatial signature of the signal received at the k -th UE after reflection from the n -th IRS.

Note that the vectors $\mathbf{u}_{k,n}$, \mathbf{u}_n and \mathbf{v}_n are defined by the system geometry whereas the optimization variables are the matrices $\mathbf{\Theta}_n$. However, for simplicity in the analysis, in the following we define the optimization vectors

$$\tilde{\mathbf{u}}_n = \mathbf{\Theta}_n \mathbf{u}_n. \quad (7)$$

In particular the ℓ -th entry of $\tilde{\mathbf{u}}_n$ is given by $\tilde{u}_{n,\ell} = \frac{1}{\sqrt{L}} e^{j\psi_{n,\ell}}$. The optimal IRS phase shifts satisfy

$$\mathbf{\Psi}^{\text{opt}} = \arg \min \|\mathbf{H}^{-1}\|_F^2 = \arg \min \text{Tr}\{(\mathbf{H}\mathbf{H}^H)^{-1}\} \quad (8)$$

A particular choice of the optimization vectors is $\tilde{\mathbf{u}}_n = \mathbf{u}_{k,n}$ for some k . If this choice is satisfied for all $n = 1, 2$, it will be called a *IRS-user association*.

We are interested in the asymptotic scenario in which i) the BS is equipped with infinitely many antennas ($M \rightarrow \infty$), and ii) the number of IRS elements tends to infinity ($L \rightarrow \infty$).

- The first condition implies that, for angularly separated IRSs, $\mathbf{v}_1^H \mathbf{v}_2 \rightarrow 0$. Practically, it says that the BS is able to form two non-interfering beams towards the two IRSs. It also allows to simplify the objective function in the optimization problem:

$$\|\mathbf{H}^{-1}\|_F^2 = \text{Tr}\{(\mathbf{H}\mathbf{H}^H)^{-1}\} = \text{Tr}\{(\mathbf{M}\mathbf{M}^H)^{-1}\} \quad (9)$$

where $[\mathbf{M}]_{k,n} = a_{k,n} \mathbf{u}_{k,n}^H \tilde{\mathbf{u}}_n$.

- Similarly, the second condition implies that, for angularly separated UEs, $\mathbf{u}_{1,n}^H \mathbf{u}_{2,n} \rightarrow 0$, $n = 1, 2$. From the physical point-of-view, it means that the power that is reflected towards UE 1 does not interfere with UE 2, and vice versa. It allows for a simple geometrical interpretation of the communication model, as we will show in the next section.

Notice that the optimization vectors satisfy an equimodular condition, i.e., $|\tilde{\mathbf{u}}_{n,\ell}| = \frac{1}{\sqrt{L}}$ for all ℓ . In the next section, we will solve the above optimization problem *without* considering the equimodular condition. The result will give sufficient condition for IRS-user association to be optimal.

3 The unconstrained optimization problem

As already said, for $L \rightarrow \infty$, $\mathbf{u}_{1,n}^H \mathbf{u}_{2,n} \rightarrow 0$, $n = 1, 2$, provided that all users are angularly separated. Thus, for IRS n , the space of useful signal is the bidimensional space spanned by the orthonormal basis $\{\mathbf{u}_{1,n}, \mathbf{u}_{2,n}\}$. So, we can write

$$\tilde{\mathbf{u}}_n = \cos \theta_n \cos \vartheta_n \mathbf{u}_{1,n} + \cos \theta_n \sin \vartheta_n \mathbf{u}_{2,n} + \sin \theta_n \tilde{\mathbf{u}}_n^\perp \quad (10)$$

where $\tilde{\mathbf{u}}_n^\perp$ is a unit-vector orthogonal to the useful signal space. By substituting (10) in (9) we get

$$\|\mathbf{H}^{-1}\|_F^2 = \frac{1/\gamma_1^2(\mu_{11}^2 \kappa_1^2 + \mu_{21}^2 \sigma_1^2) + 1/\gamma_2^2(\mu_{12}^2 \kappa_2^2 + \mu_{22}^2 \sigma_2^2)}{(\mu_{11} \mu_{22} \kappa_1 \sigma_2 - \mu_{12} \mu_{21} \sigma_1 \kappa_2)^2} \quad (11)$$

where we defined $\gamma_n = \cos \theta_n$, $\kappa_n = \cos \vartheta_n$, $\sigma_n = \sin \vartheta_n$ and $\mu_{k,n} = |a_{k,n}|$. Now, we solve the optimization problem without considering the equimodular condition on $\tilde{\mathbf{u}}_n$. First, the minimum of $\|\mathbf{H}^{-1}\|_F^2$ is obtained for $\gamma_1 = \gamma_2 = 1$, i.e., $\tilde{\mathbf{u}}_n$ belongs to the useful signal space, a pretty obvious fact. To minimize $\|\mathbf{H}^{-1}\|_F^2$ with respect to ϑ_n , $n = 1, 2$, we set $\nabla \|\mathbf{H}^{-1}\|_F^2 = \mathbf{0}$. By writing $\|\mathbf{H}^{-1}\|_F^2 = f_1/f_2$ where f_1 and f_2 are, respectively, the numerator and the denominator of (11), we have

$$\frac{\partial \|\mathbf{H}^{-1}\|_F^2}{\partial \vartheta_n} = 0 \iff f_2 \frac{\partial f_1}{\partial \vartheta_n} = f_1 \frac{\partial f_2}{\partial \vartheta_n}.$$

After a little bookkeeping, and defining $\zeta_n(\sigma_n^2) \triangleq \mu_{1,n}^2 + (\mu_{2,n}^2 - \mu_{1,n}^2) \sigma_n^2$ we obtain the following two equations, $n = 1, 2$:

$$\mu_{1,n} \mu_{2,\bar{n}} \kappa_1 \kappa_2 (\mu_{1,\bar{n}}^2 + \zeta_n) + \mu_{1,\bar{n}} \mu_{2,n} \sigma_1 \sigma_2 (\mu_{2,\bar{n}}^2 + \zeta_n) = 0 \quad (12)$$

where $\bar{n} = 3 - n$. The above equations are satisfied if $\sigma_1 \sigma_2 = 0$ and $\kappa_1 \kappa_2 = 0$, giving the two stationary points $(\vartheta_1, \vartheta_2) = (0, \pi/2)$ and $(\vartheta_1, \vartheta_2) = (\pi/2, 0)$. They correspond to two IRS-user associations: the first point associates UE 1 to IRS 1 and UE 2 to IRS 2, while the second assigns UE 2 to IRS 1 and UE 1 to IRS 2. However, there might be another stationary point, depending on the $\mu_{k,n}$ values. By equating the two LHSs of the pair of equations (12), we obtain:

$$c_1 \zeta_1 + c_2 \zeta_2 + c_3 \zeta_1 \zeta_2 = 0 \quad (13)$$

where $c_1 = \mu_{11}^2 \mu_{21}^2 (\mu_{22}^2 - \mu_{12}^2)$, $c_2 = \mu_{12}^2 \mu_{22}^2 (\mu_{11}^2 - \mu_{21}^2)$ and $c_3 = \mu_{11}^2 \mu_{22}^2 - \mu_{21}^2 \mu_{12}^2$. If $c_1 c_2 > 0$, all terms in (13) have the same sign, and since $\zeta_n > 0$, there is no solution to (13), so that one of the two IRS-user associations is the global optimum. If instead $c_1 c_2 < 0$, (13) describes a hyperbola in the plane (ζ_1, ζ_2) , which can be parametrized through standard techniques to obtain

$$\zeta_1 = \frac{1}{t} \tan\left(\frac{\alpha + \frac{\pi}{2}}{2}\right) - \frac{c_2}{c_3}; \quad \zeta_2 = \frac{1}{t} \tan\left(\frac{\alpha - \frac{\pi}{2}}{2}\right) - \frac{c_1}{c_3}$$

with $t = \sqrt{\frac{c_3^2}{-c_1 c_2}}$ and α a parameter. Define $b_n = \min\{\mu_{1,n}^2, \mu_{2,n}^2\}$ and $B_n = \max\{\mu_{1,n}^2, \mu_{2,n}^2\}$, $n = 1, 2$. Since $b_n \leq \zeta_n \leq B_n$, we have the following bounds on α :

$$2 \arctan b'_1 - \frac{\pi}{2} \leq \alpha \leq 2 \arctan B'_1 - \frac{\pi}{2} \quad (14)$$

$$2 \arctan b'_2 + \frac{\pi}{2} \leq \alpha \leq 2 \arctan B'_2 + \frac{\pi}{2} \quad (15)$$

where, for $n = 1, 2$, $b'_n = t(b_n + \frac{c_n}{c_3})$ and $B'_n = t(B_n + \frac{c_n}{c_3})$. The stationary point can only be present if the above intervals intersect. This happens if at least one of these two conditions holds:

$$-\frac{1}{b'_2} \leq b'_1 \leq -\frac{1}{B'_2}, \quad -\frac{1}{b'_1} \leq b'_2 \leq -\frac{1}{B'_1} \quad (16)$$

A thorough analysis of the above inequalities brings up to the following proposition, which completely describes when the IRS-user association is optimal.

Proposition 1. *The function $\|\mathbf{H}^{-1}\|_F^2 = \text{Tr}\{(\mathbf{M}\mathbf{M}^H)^{-1}\}$ achieves its minimum for a IRS-user association when one of the following two conditions are satisfied.*

$$1. \frac{1}{\mu_{11}^2} + \frac{1}{\mu_{22}^2} < \frac{1}{\mu_{12}^2} + \frac{1}{\mu_{21}^2} \text{ and}$$

$$\mu_{11}^2 \mu_{22}^2 (\mu_{22}^2 + \mu_{21}^2) (\mu_{11}^2 + \mu_{12}^2) > \mu_{12}^2 \mu_{21}^2 (\mu_{11}^2 + \mu_{22}^2)^2 \quad (17)$$

$$2. \frac{1}{\mu_{11}^2} + \frac{1}{\mu_{22}^2} > \frac{1}{\mu_{12}^2} + \frac{1}{\mu_{21}^2}, \text{ and}$$

$$\mu_{12}^2 \mu_{21}^2 (\mu_{22}^2 + \mu_{21}^2) (\mu_{11}^2 + \mu_{12}^2) > \mu_{11}^2 \mu_{22}^2 (\mu_{12}^2 + \mu_{21}^2)^2 \quad (18)$$

Proof. Consider the first case. A straightforward computation shows that, if (17) is satisfied, neither condition in (16) is satisfied, so there is no solution to (13). Note also that (17) corresponds to stating that the Hessian of $\|\mathbf{H}^{-1}\|_F^2$ in the stationary point $(\vartheta_1, \vartheta_2) = (0, \pi/2)$ is positive definite. This implies that $(\vartheta_1, \vartheta_2) = (0, \pi/2)$ is the global minimum of $\|\mathbf{H}^{-1}\|_F^2$. The same reasoning can be applied to the second case, with the global minimum being $(\vartheta_1, \vartheta_2) = (\pi/2, 0)$. \square

4 The constrained optimization problem

Whenever the conditions of Proposition 1 are not satisfied, the unconstrained global minimum of $\|\mathbf{H}^{-1}\|_F^2 = \text{Tr}\{(\mathbf{M}\mathbf{M}^H)^{-1}\}$ is not an IRS-user association. However, since we haven't considered the equimodular constraint, such global minimum is not necessarily feasible. In this section, we consider the equimodular constraint and we particularize the problem as follows. We suppose that a fraction z_n of the elements of IRS n are properly configured to steer the signal toward UE n , while the remaining elements cooperate to point toward the other UE. While it is a particular IRS configuration scenario, in the asymptotic regime, it can be seen as a reasonable, if not optimal, strategy.

Thus, the optimization problem becomes:

$$\min_{z_1, z_2} \text{Tr}\{(\mathbf{M}\mathbf{M}^H)^{-1}\}, \text{ s.t. } 0 \leq z_n \leq 1, n=1,2 \quad \text{where} \quad (19)$$

$$\mathbf{M} = \begin{bmatrix} \mu_{11}z_1 e^{j\omega_{11}} & \mu_{12}(1-z_2)e^{j\omega_{12}} \\ \mu_{21}(1-z_1)e^{j\omega_{21}} & \mu_{22}z_2 e^{j\omega_{22}} \end{bmatrix} \quad (20)$$

where $\{\omega_{kn}\}_{k,n=1}^2$ are arbitrary phases satisfying $\omega_{11} + \omega_{22} - (\omega_{21} + \omega_{12}) = \pi$ to maximize the modulus of the determinant of \mathbf{M} .

Proceeding analogously as in the previous section, we can find the conditions satisfied by stationary points. Based on these conditions, stationary points can be found numerically by standard methods. In all tested cases, it seems that there exists a single stationary point, although we do not have a proof. The following proposition, whose proof is omitted for lack of space, helps in the numerical search.

Proposition 2. Let $A_0 = \frac{\mu_{21}\mu_{12}}{\mu_{21}\mu_{12} + \mu_{11}\mu_{22}}$, $A_1 = \frac{\mu_{21}^2}{\mu_{21}^2 + \mu_{11}^2}$ and $A_2 = \frac{\mu_{12}^2}{\mu_{12}^2 + \mu_{22}^2}$. If $\mu_{11}\mu_{12} > \mu_{21}\mu_{22}$, any stationary point (z_1^*, z_2^*) will satisfy $A_1 < z_1^* < A_0$ and $A_0 < z_2^* < A_2$. If instead $\mu_{11}\mu_{12} < \mu_{21}\mu_{22}$, then $A_0 < z_1^* < A_1$ and $A_2 < z_2^* < A_0$.

If a stationary point is found, the function value in this point must be compared with the value for the best IRS-user association obtained in the unconstrained optimization.

5 Simulation results

In this section, we show some simulation results in which we generate with different statistics the values of $[\mu_{k,n}]_{k,n=1}^2$ and compute by Monte-Carlo simulation the probability that IRS-user association is not optimal, denoted p_{NO} in the following. In Fig. 2, we show three different curves that correspond to the following statistics.

Solid curves correspond to the case $\mu_{1,1}, \mu_{2,1} \sim U[0,1]$ while $\mu_{1,2}, \mu_{2,2} \sim U[0,K]$, i.e., IRS 2 is on the average favored w.r.t. IRS 1 ("Column unbalance" scenario). In this case, p_{NO} is always low, and slightly decreasing with increasing values of the unbalance K .

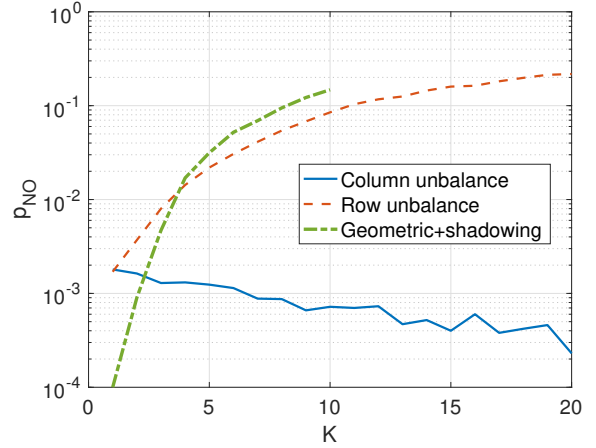


Figure 2. p_{NO} in the different scenarios.

Dashed curves correspond to the case $\mu_{1,1}, \mu_{1,2} \sim U[0,1]$ while $\mu_{2,1}, \mu_{2,2} \sim U[0,K]$, i.e., user 2 is on the average favored w.r.t. user 1 ("Row unbalance" scenario). As it can be seen, for increasing the unbalance K , IRS-user association is less likely to be optimal, with the maximum of p_{NO} being about 0.2 for $K = 20$. We explain this behavior by noticing that, for large unbalance, both IRSs are needed to reach the weaker user, in order to obtain the same SNR for both users.

Finally, dash-dotted curves correspond to the case of the model in Section 2. In particular, we suppose that, in the plane (x, y) , the BS is at point $(-1.5, 1)$, the two IRSs fixed at positions $(-0.5, 0)$ and $(0.5, 0)$, while the two users are uniformly distributed within circles of radius 1 and K , respectively, centered at the origin ("Geometric" scenario). Moreover, we have introduced an independent lognormal shadowing coefficient on all taps, with standard deviation 1 dB. The behavior of this scenario is similar to the first one and yields a value of p_{NO} smaller than 0.14 for $K \leq 10$.

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