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Solution of Inverse Source Problems with Distributed Spherical Harmonics Expansions

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Abstract

Inverse source solutions for field transformations and diagnostics are mostly working with surface current densities on appropriately defined Huygens surfaces around the test object. This gives excellent modeling flexibility, but the handling of the discretized surface current representation requires also substantial computational effort. Distributed spherical harmonics expansions of low order, for example based on a solution space partitioning as used in the multilevel fast multipole method, have somewhat reduced modeling flexibility, but they can save considerable computational effort, and they are, thus, an excellent choice of expansion functions for many practical inverse source problems. We discuss different forms of distributed spherical harmonics expansions comprising purely scalar spherical modes and vector spherical modes. Moreover, we discuss how the different surface source expansions consisting of electric and magnetic surface currents with Love condition or without, or consisting of directive Huygens radiators can be related to corresponding distributed spherical harmonics expansions.

1 Introduction

Electromagnetic field transformations, e.g., near-field (NF) far-field (FF) transformations, are routinely performed by utilizing modal-expansion based algorithms and corresponding spherical, cylindrical, or planar measurement arrangements, which support the orthogonality of the modes and, thus, the effectiveness of the transformation algorithms [1, 2]. Considerably more flexibility for the cost of more computational effort is provided by inverse source algorithms, which work with surface sources distributed on a Huygens surface around the device under test (DUT), where our interest is here in fully three-dimensional approaches such as found in [3, 4, 5, 6, 7]. In order to limit the computational demands of these methods, they are often accelerated by fast integral algorithms, where in particular the fast multipole method (FMM) [8] and the multilevel fast multipole method (MLFMM) have been proven to be very useful [4, 5, 9].

An important question related to the inverse source formulations is how to choose an appropriate set of surface sources. According to the uniqueness and Huygens principles [10, 11, 12], it is well known that electric (or magnetic) surface current densities alone, or combinations of both of them can be chosen to correctly represent the fields of the DUT. However, it is also known that electric or magnetic surface current densities alone (often called single-source formulation) lead to a badly conditioned system of equations, which is hard to solve [6, 13]. Therefore, the common choice are combinations of electric and magnetic surface current densities, which are often even combined with an explicit imposition of a Love or zero-field condition in order to obtain the so-called Love surface current densities [6, 7, 11, 14], which are directly related to the tangential electric and magnetic fields on the Huygens surface. The Love surface current densities are useful for diagnostic inspections, however, their enforcement during the solution process is certainly not recommended [2, 15, 16]. Instead, the Love currents or the fields can easily be computed in a post-processing step, and, if desired, a reduction of the effective number of unknowns during the solution process can very efficiently be achieved by forming directive Huygens radiators based on a strong or weak-form imposition of an appropriate impedance boundary condition [13, 17].

As an alternative to spatial surface current density representations, distributed spherical harmonics expansions have been presented in [18, 2] as equivalent sources for the representation of the DUT fields. Following the introduction of such expansions within the MLFMM in [19], scalar spherical harmonics have been used to represent the Cartesian components of the fields, but TE- and TM-vector spherical harmonics [20] can of course also be used for this purpose. Interesting in this respect is the question of how considerations on uniqueness, Love condition, etc. translate to distributed source representations with spherical harmonics.

In Section 2, an inverse source formulation with surface current representation is introduced and then specialized to work with distributed spherical harmonics expansions, where the derivation is based on the propagating plane-wave representation of the Green's functions as known from the MLFMM. Scalar spherical harmonics and vector spherical harmonics are considered and their relation to electric and magnetic surface current densities is discussed. In Section 3, inverse source solutions with distributed spherical harmonics expansions are presented and compared to solutions with surface current density expansions, before some conclusions are drawn in Section 4.

2 Inverse Source Formulations with Distributed Spherical Harmonics Expansion

An inverse source formulation starts advantageously with spatial electric surface current densities $\mathbf{J}_A$ and magnetic surface current densities $\mathbf{M}_A$ defined on a Huygens surface A enclosing the volumetric support of the DUT in the form of [18]

$$U(\mathbf{r}_m) = \iiint_{V_w} \mathbf{w}(\mathbf{r}-\mathbf{r}_m)\cdot \iint_A \left[\bar{\mathbf{G}}_J^E(\mathbf{r},\mathbf{r}')\cdot \mathbf{J}_A(\mathbf{r}') + \right. \\ \left. + \bar{\mathbf{G}}_M^E(\mathbf{r},\mathbf{r}')\cdot \mathbf{M}_A(\mathbf{r}')\right] \mathrm{d}a'\mathrm{d}v, \quad (1)$$

where the field observations in form of the voltages $U(\mathbf{r}_m)$ at the observation locations $\mathbf{r}_m$ are weighted by a probing antenna with a spatial weighting function $\mathbf{w}$ over the probe volume V_w. A time factor $\mathrm{e}^{j\omega t}$ with angular frequency ω is assumed throughout and suppressed. $\bar{\mathbf{G}}_J^E$ and $\bar{\mathbf{G}}_M^E$ are the pertinent dyadic free-space Green's functions, respectively. With a discretized representation of the surface sources according to

$$\mathbf{J}_A(\mathbf{r}) = \sum_p J_p \boldsymbol{\beta}_p(\mathbf{r}), \quad \mathbf{M}_A(\mathbf{r}) = \sum_q M_q \boldsymbol{\beta}_q(\mathbf{r}), \quad (2)$$

e.g., with divergence conforming vector basis functions $\boldsymbol{\beta}_p$ and $\boldsymbol{\beta}_q$, the coefficients J_p and M_q become the unknowns of a discrete linear inverse problem, which can be solved if sufficiently many field observations are available.

Based on the Gegenbauer propagating plane-wave representation of the scalar Green's function of the Helmholtz equation [21, 18]

$$\frac{\mathrm{e}^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \approx \oiint \mathrm{e}^{-j\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_m)} \mathrm{e}^{j\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r}'_s)}\, T_L(\mathbf{k},\mathbf{r}_m-\mathbf{r}'_s)\, d\hat{k}^2 \quad (3)$$

with a diagonal plane-wave translation operator $T_L(\mathbf{k},\mathrm{X})$ [18], the weighted radiation operator (1) can be converted into

$$U(\mathbf{r}_m) \approx \frac{-j}{4\pi} \oiint \left[\tilde{\mathbf{w}}(-\mathbf{k})\cdot T_L(\mathbf{k},\mathbf{r}_m-\mathbf{r}'_s)\left(\tilde{\mathbf{J}}(\mathbf{k})+\tilde{\mathbf{M}}(\mathbf{k})\right)\right] d\hat{k}^2, \quad (4)$$

where the accuracy is error-controllable by an appropriate choice of L [21]. The tilde indicates here spectral quantities defined over the Ewald sphere of propagating plane waves, which can be found in [18, 13] and which are computed by evaluating the corresponding Fourier-type integrals over the spatial quantities with respect to the probe reference location $\mathbf{r}_m$ or with respect to a source reference location $\mathbf{r}'_s$, respectively. Since these integrals need to be pre-computed with respect to the smallest source boxes within the MLFMM hierarchy and stored in memory in the set-up phase of an efficient inverse source solver, it appears, of course, very attractive to avoid these steps and work directly with an expansion of the spectral representation of the source quantities. Following [18, 19], the combined plane-wave spectra of the electric and magnetic current densities of the finest-level source boxes are represented according to

$$\tilde{\mathbf{J}}(\mathbf{k}) + \tilde{\mathbf{M}}(\mathbf{k}) = \boldsymbol{T}(\mathbf{k}) \sum_{n=0}^{M_L} \sum_{m=-n}^{n} \mathbf{f}_{nm} \mathrm{Y}_{nm}(\mathbf{k}), \quad (5)$$

where $\mathbf{f}_{nm}$ are the new expansion coefficients in Cartesian components and $\mathrm{Y}_{nm}(\mathbf{k})$ are the scalar spherical harmonics of degree m and order n [19, 21]. Since the MLFMM-based algorithm works with transverse field components on the Ewald sphere, the Cartesian vector components are converted into ϑ- and φ-components by the matrix operator $\boldsymbol{T}$. The degree M_L of the expansion must be chosen according to the electrical size of the source boxes and is commonly rather small (often below 10). This was actually the reason for choosing a scalar expansion of Cartesian components in [19]. For small degrees, the required number of scalar expansion coefficients can be smaller than the required number of vector expansion coefficients with TE- and TM-vector spherical harmonics according to

$$\tilde{\mathbf{J}}(\mathbf{k}) + \tilde{\mathbf{M}}(\mathbf{k}) = \sum_{n=1}^{M_L+1} \sum_{m=-n}^{n} \left[c_{nm}^{TM}\mathbf{n}_{nm}(\mathbf{k}) + c_{nm}^{TE}\mathbf{m}_{nm}(\mathbf{k})\right], \quad (6)$$

where the summation ranges now from $n = 1$ to $M_L + 1$. The $\mathbf{n}_{nm}(\mathbf{k})$ and $\mathbf{m}_{nm}(\mathbf{k})$ are the transverse angular parts of the TM- and TE-vector spherical harmonics [20] and c_{nm}^{TM} and c_{nm}^{TE} are the corresponding expansion coefficients. For larger degrees, the vector expansion needs fewer expansion coefficients than the scalar expansion and it has the advantage that every field mode corresponds to a valid solution of Maxwell's equations. With this in mind, we can relate the electric and magnetic current sources of every field mode to the surface current densities in our original representation in (1) and we can transfer the discussion about the appropriate choice of surface current density expansion to the case of the vector spherical harmonics. Restricting ourselves to the case of just dipoles, it is immediately clear that we may work only with those vector spherical modes which correspond to dipole currents in parallel to our Huygens surface and we may construct for instance Huygens radiators as found in [13].

3 Numerical Results

In order to demonstrate the behavior of inverse source solutions with different distributed spherical harmonics expansions, we consider the scattering problem of a corner reflector as shown in Fig. 1, which has, e.g., already been considered in [18]. The regularly sampled NF data on a plane parallel to and in front of the aperture of the reflector and the FF reference data have been obtained by

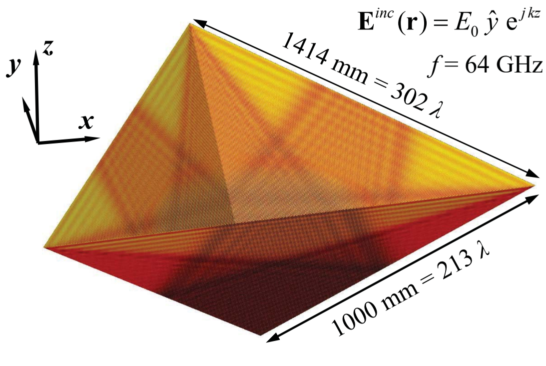
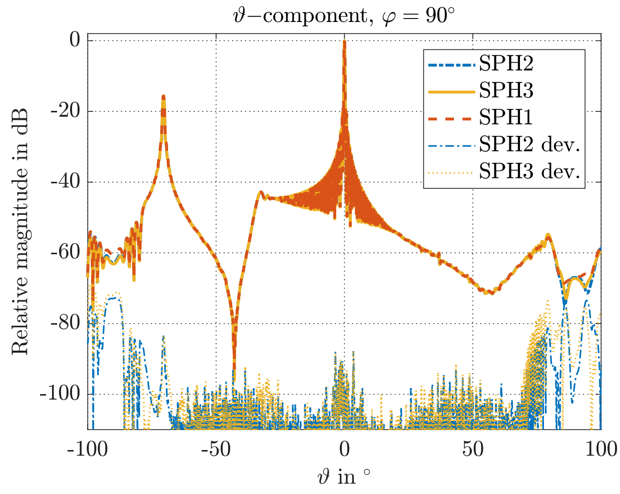


Figure 1. Corner reflector with electric surface current density from method of moments solution for an incident plane wave with the indicated electric field and frequency.

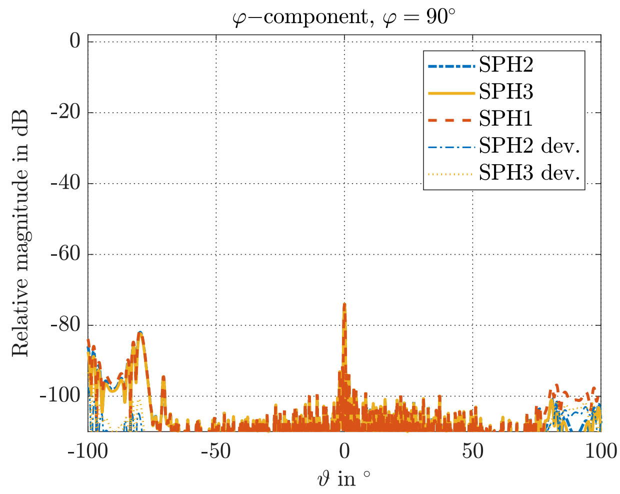
Table 1. Numerical performance of the various inverse source solutions (JM: solver with electric and magnetic surface current densities according to (2), JH: solver with Huygens radiator expansion (see [13]), J: solver with electric surface currents only, SPH1: expansion according to (5), SPH2: expansion according to (6), SPH3: expansion with directive vector multipoles on the basis of (6), n_{it}: number of iterative solver iterations, t_{inv}: iterative solver time, t_{tot}: total solution time (including setup), RAM: random access memory.

solver	n_{it}	t_{inv}/s	t_{tot}/s	RAM/GB
JM	84	4989	5171	48
JH	85	4382	4552	48
J	176	9129	9329	53
SPH1	80	3664	3740	39
SPH2	79	3747	3823	39
SPH3	74	3361	3438	38

a methods of moments surface integral equation solution [19] for a plane wave incident normally with respect to the aperture. The inverse source problem with 6 487 202 NF samples was solved with an iterative inverse source solver on the basis the normal error system of normal equations [16] with different source expansions. The computational results obtained with an Intel(R) Core(TM) i7-4820K CPU @ 3.7 GHz with four cores are summarized in Table 1, where also the utilized acronyms for the different source expansions and the considered performance parameters are explained. Solutions with surface current expansions related to (2) provide for the full localization and diagnostics performance, but due to the handling of the basis functions and the mesh, they do need a certain extra effort, e.g., for the pre-computation and storage of their k-space representations. The more or less meshless expansions according to (5) and (6) require less computational resources, but have also somewhat reduced localization performance.



(a)



(b)

Figure 2. Normalized FF cut of bistatic scattering magnitude of corner reflector in Fig. 1 (see Table 1 for the acronyms, dev.: deviation with respect to the SPH1 result).

The SPH3 expansion works with a directive spherical vector multipole expansion similar to the concept of the Huygens radiator (JH in Table 1) and needs, thus, in comparison to SPH2 only half the number of unknowns to represent a valid set of equivalent sources without loss of accuracy. The FF bistatic scattering results in Fig. 2 show that all distributed spherical harmonics based expansions show very good agreement well below the accuracy of the not shown reference data (see, e.g., [18] for a comparison with reference data).

4 Conclusion

Distributed spherical harmonics expansions of the radiation or scattering fields of test objects, which need to be characterized based on field observations in some distance away from the objects, have been discussed. Such expansions still provide considerable modeling flexibility, but the computational effort of related inverse source solvers is clearly smaller than of field representations with spatial current density representations. The benefits of scalar and vector expansions of the radiation fields have been discussed and it was highlighted that further variations of the expansions can be of benefit based on the applicable uniqueness considerations.

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