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Mathematical modeling of electromagnetic fields: From classical functions to Schwartz distributions to Colombeau generalized functions

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Extended abstract

Functions that map space-time coordinates to real numbers are ill-suited to model electromagnetic fields. Indeed, some common physical derivations yield mathematical objects that cannot be represented by such usual functions. In this work, we propose to shed light on some ambiguities that arise in commonly seen derivations, and present a landscape of the mathematical tools which can be used to solve these issues, with particular emphasis on applied electromagnetism.

For example, Green's functions offer a convenient way to represent electromagnetic fields and derive some of their properties. However, the definition of these functions relies on an inhomogeneous partial differential equation involving the Dirac δ function. This function also appears in the Fourier transforms of common signals, such as a constant or the ubiquitous complex-valued exponential. Since any real-valued function with point support has zero Lebesgue integral, the Dirac δ function cannot be represented as a function mapping space-time coordinates to real numbers. Moreover, while the Dirac δ function can be represented as a measure, its derivatives cannot. Formally, they are Schwartz distributions.

Schwartz distributions [1] can be seen as a refined physical model of fields. Indeed, the evaluation of some field f at the coordinates $(t, \mathbf{x})$ assumes knowledge of f at exactly t and $\mathbf{x}$. This is an idealization of the measurement process, where we usually take a time-average of a measurement occurring over a finite region of space. The measurement can thus be represented by a test function (i.e., a smooth function with compact support), and the field as a map between test functions and numbers. The modeling of electromagnetic fields as Schwartz distributions not only clarifies expressions involving the Dirac δ distribution, its derivatives, Green functions, convolutions, and frequency domain techniques, but it also allows to easily compute the multipole expansion of the electric or magnetic fields. Moreover, the multipole expansion convergence region can be explicitly computed.

On the other hand, by setting aside the notion of point evaluation of a field, Schwartz distributions cannot be multiplied, integrals of Schwartz distributions must be handled with care, and nonlinear transformations of Schwartz distributions are ill-defined. For example, it is not formally possible to relate fields on the boundary of a volume to the fields inside, because we cannot apply Green's formula to Schwartz distributions. Luckily, this issue can be resolved by modeling the fields as Colombeau generalized functions [2]. Any function or Schwartz distribution can be seen as a Colombeau generalized function. Moreover, these functions represent more general objects, such as products of Schwartz distributions. In practice, Colombeau generalized functions are represented by smooth functions, making their manipulations straightforward. Furthermore, the construction of these generalized functions yields the notions of microscopic and macroscopic equality of fields [3], which find physical interpretations when applying nonlinear transforms to fields, such as for the electromagnetic time-reversal cavity. Together with Schwartz distributions, Colombeau generalized functions offer a formal and straightforward mathematical setting for theoretical and applied analyses of electromagnetic fields.

References

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