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Surface waves in an open circular waveguide filled with inhomogeneous chiral media

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Abstract

Propagation of surface waves in a radially inhomogeneous chiral waveguide is considered. The setting is reduced to a boundary eigenvalue problem for the longitudinal components of the electromagnetic field in Sobolev spaces. To find the solution, variational formulation is used. The variational problem is reduced to the analysis of an operator-valued function. Discreteness of the spectrum is proved and distribution of the characteristic numbers of the operator-valued function on the complex plane is determined. The results of numerical modeling of the spectrum of propagating surface waves in an open chiral waveguide are presented.

1 Introduction

We consider the wave propagation in a dielectric rod filled with an inhomogeneous chiral media. This is an example of an open waveguide for which the existence of normal waves and the distribution of the wave spectra on the complex plane remain unsolved problems.

Investigations of waveguides filled with chiral media have been a subject of intense studies since the late 1970s (see e.g. [3, 4] and references therein). However, numerous papers in this field report only the results of numerical modeling; rigorous proofs revealing the existence and localizaiton of the wave spectra have never been obtained, to the best of our knowledge.

This circumstance has served as a driving force to accomplish this study aimed at creating foundations of a rigorous theory of the wave propagation in chiral waveguides and a justified numerical method for their calculation.

Generally, the determination of the spectrum of normal waves in open structures is reduced [1, 2] to nonselfadjoint boundary eigenvalue problems for the systems of Helmholtz equations with piecewise constant coefficients and the transmission conditions containing the spectral parameter. On the discontinuity lines (surfaces) additional conditions are stated called the transmission conditions. In classical settings applicable in particular for shielded waveguides with homogeneous dielectrics, the spectral parameter is present in the equations and not in the transmission conditions (and conditions at infinity for open structures) which results in eigenvalue problems for selfadjoint operators. For inhomogeneous media filling the waveguide, the spectral parameter enters both the equation and the transmission conditions, often in a nonlinear manner, yielding nonselfadjoint eigenvalue problems. The latter are reduced to the analysis of spectra of operator-valued functions nonlinear with respect to the spectral parameter which consitutes the only possible and natural mathematical way to investigate oscillations and waves in open inhomogeneously filled waveguides.

An approach proposed in [1] employing the reduction of eigenvalue problems to the determination of characteristic numbers of operator-valued functions and operator pencils considered in Sobolev spaces was developed by Y. Smirnov in [5, 6, 7] (see also [8, 9]). General theory of polynomial operator-functions called operator pencils is sufficiently well elaborated. A fundamental work by Keldysh [10] pioneered investigation of nonselfadjoint polynomial pencils.

The method of operator pencils has proved to be an efficient technique for the analysis of the wave propagation in open and shielded waveguides. Operator pencils were applied to the analysis of electromagnetic problems in [11, 12, 13].

Recently, in connection with the progress in polymer technologies, new synthesized chiral materials have appeared, which has enhanced the interest of researchers to this field. [14, 15, 16, 17] and caused a necessity to elaborate adequate mathematical and numerical methods for the analysis of waves in chiral waveguides.

In these problems, however, we have to analyze not operator pencils (as in [7]), but operator-valued functions. Nevertheless, it is possible to apply and develop the previously created techniques to study the properties of the operator–valued function in sufficient detail and obtain the results describing their spectrum. In this work, the discreteness of the spectrum of normal waves in chiral waveguides is proved and the distribution of characteristic numbers on the complex plane is presented.

Note that we consider waves that decrease at a distance from the waveguide (we impose the corresponding conditions at infinity). Other types of waves are not considered. This approach was used in our earlier works to study shielded waveguide structures [18, 19, 20]. The results of a numerical study of the spectrum of propagating surface waves in an open chiral waveguide are presented; they have been obtained using a specially created numerical method employing the developed variational setting.

2 Statement of the problem

Consider the three-dimensional space $\mathbb{R}^3$ with the cylindrical coordinate system $O\rho\varphi z$. The space is filled with an isotropic source-free medium having permittivity $\varepsilon = \varepsilon_0 \equiv const$ and permeability $\mu = \mu_0 \equiv const$, where ε_0 and μ_0 are permittivity and permeability of vacuum. A waveguide with a cross-section

$$\Sigma := \{(\rho, \varphi, z) : r_0 \leqslant \rho \leqslant r, 0 \leqslant \varphi < 2\pi\}$$

and a generating line parallel to the axis Oz is placed in $\mathbb{R}^3$.

The cross section of the waveguide by a plane perpendicular to its axis consists of two concentric circles of radii r_0 and r (see Fig. 1): r is the radius of the internal (perfectly conducting) cylinder and $r - r_0$ is the thickness of the external (dielectric) cylindrical shell. The geometry of the problem is shown in Fig. 1.

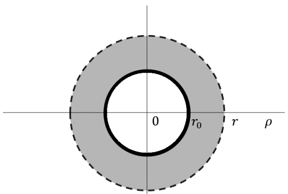

Figure 1. Geometry of the problem.

The problem on normal waves in a waveguide structure reduces to finding nontrivial running-wave solutions to the homogeneous system of Maxwell equations, i.e., solutions with the dependence of the from $e^{i\gamma z}$ on the coordinate z along which the structure is regular,

$$\begin{cases} \operatorname{rot}\mathbf{H} = -i\omega\widetilde{\varepsilon}\mathbf{E} - \omega\widetilde{\chi}\mathbf{H}, \\ \operatorname{rot}\mathbf{E} = i\omega\widetilde{\mu}\mathbf{H} - \omega\widetilde{\chi}\mathbf{E}, \end{cases} \tag{1}$$

$$\begin{aligned} \mathbf{E} &= \big(E_\rho(\rho)\,\mathbf{e}_\rho + E_\varphi(\rho)\,\mathbf{e}_\varphi + E_z(\rho)\,\mathbf{e_z}\big)\,e^{i\gamma z}, \\ \mathbf{H} &= \big(H_\rho(\rho)\,\mathbf{e}_\rho + H_\varphi(\rho)\,\mathbf{e}_\varphi + H_z(\rho)\,\mathbf{e_z}\big)\,e^{i\gamma z}, \end{aligned} \tag{2}$$

with the boundary conditions for the tangential electric components on perfectly conducting surfaces ($\rho = r_0$)

$$E_\varphi\big|_{\rho=r_0} = 0, \quad E_z\big|_{\rho=r_0} = 0, \tag{3}$$

transmission conditions for tangential electric and magnetic components on the surfaces of "discontinuity" of permittivity and permeability ($\rho = r$)

$$\begin{aligned} [E_\varphi]\big|_{\rho=r} = 0, \quad [E_z]\big|_{\rho=r} = 0, \\ [H_\varphi]\big|_{\rho=r} = 0, \quad [H_z]\big|_{\rho=r} = 0, \end{aligned} \tag{4}$$

the condition of the finiteness of energy

$$\int_{r_0}^{\infty} (\widetilde{\varepsilon}|\mathbf{E}|^2 + \widetilde{\mu}|\mathbf{H}|^2)d\rho < \infty, \tag{5}$$

and the radiation condition at infinity: the electromagnetic field decays as $o(\rho^{-1/2})$ for $\rho \to \infty$.

Permittivity, permeability, and chirality in the whole space have the form

$$\begin{aligned} \widetilde{\varepsilon} &= \begin{cases} \varepsilon(\rho),\ r_0 \le \rho \le r, \\ \varepsilon_0,\ \rho > r. \end{cases}, \\ \widetilde{\mu} &= \begin{cases} \mu(\rho),\ r_0 \le \rho \le r, \\ \mu_0,\ \rho > r. \end{cases}, \\ \widetilde{\chi} &= \begin{cases} \chi,\ r_0 \le \rho \le r, \\ 0,\ \rho > r. \end{cases}, \end{aligned} \tag{6}$$

where $\varepsilon(\rho) \in C^1[r_0,\ r]$, $\min\limits_{[r_0,\ r]} \varepsilon(\rho) > \varepsilon_0$, $\mu(\rho) \in C^1[r_0,\ r]$, $\min\limits_{[r_0,\ r]} \mu(\rho) > \mu_0$ and χ is a constant.

The problem on normal waves is an eigenvalue problem for the Maxwell equations with spectral parameter γ which is the normalized propagation constant.

Rewrite system (1) in the expanded form:

$$\begin{cases} -i\gamma H_\varphi = -i\omega\widetilde{\varepsilon}E_\rho - \omega\widetilde{\chi}H_\rho, \\ i\gamma H_\rho - H_z' = -i\omega\widetilde{\varepsilon}E_\varphi - \omega\widetilde{\chi}H_\varphi, \\ \dfrac{1}{\rho}(\rho H_\varphi)' = -i\omega\widetilde{\varepsilon}E_z - \omega\widetilde{\chi}H_z, \\ -i\gamma E_\varphi = i\omega\widetilde{\mu}H_\rho - \omega\widetilde{\chi}E_\rho, \\ i\gamma E_\rho - E_z' = i\omega\widetilde{\mu}H_\varphi - \omega\widetilde{\chi}E_\varphi, \\ \dfrac{1}{\rho}(\rho E_\varphi)' - = i\omega\widetilde{\mu}H_z - \omega\widetilde{\chi}E_z, \end{cases} \tag{7}$$

and express functions E_ρ, H_ρ, E_z, H_z via functions E_φ and H_φ from the 1st, 3rd, 4th, and 6th equations in system (1)

$$\begin{aligned} E_\rho &= \frac{\gamma}{\omega}\frac{i\widetilde{\chi}E_\varphi - \widetilde{\mu}H_\varphi}{\widetilde{\chi}^2 - \widetilde{\varepsilon}\widetilde{\mu}}, \qquad E_z = -\frac{1}{\omega\rho}\frac{\widetilde{\chi}(\rho E_\varphi)' + i\widetilde{\mu}(\rho H_\varphi)'}{\widetilde{\chi}^2 - \widetilde{\varepsilon}\widetilde{\mu}}, \\ H_\rho &= \frac{\gamma}{\omega}\frac{i\widetilde{\chi}H_\varphi + \widetilde{\varepsilon}E_\varphi}{\widetilde{\chi}^2 - \widetilde{\varepsilon}\widetilde{\mu}}, \qquad H_z = -\frac{1}{\omega\rho}\frac{\widetilde{\chi}(\rho H_\varphi)' - i\widetilde{\varepsilon}(\rho E_\varphi)'}{\widetilde{\chi}^2 - \widetilde{\varepsilon}\widetilde{\mu}}. \end{aligned} \tag{8}$$

It follows from Eqs. (8) that the normal wave field in the waveguide can be represented with the use of two scalar functions

$$u_e := iE_\varphi(\rho),\ u_m := H_\varphi(\rho). \tag{9}$$

Thus, the problem has been reduced to finding the tangential components u_e and u_m of the electric and magnetic fields. Throughout the text, $(\,\cdot\,)'$ stands for differentiation with respect to ρ.

For the tangential field components u_e and u_m we have the following eigenvalue problem: find such $\gamma \in \mathbb{C}$ for which there exist non-trivial solutions of the following system of differential equations

$$\begin{aligned} u_e'' + \widetilde{h}_e u_e' + \left(\widetilde{g}_e - \gamma^2\right) u_e &= \widetilde{f}_m (\rho u_m)' + \widetilde{k}_m u_m, \\ u_m'' + \widetilde{h}_m u_m' + \left(\widetilde{g}_e - \gamma^2\right) u_m &= \widetilde{f}_e (\rho u_e)' + \widetilde{k}_e u_e, \end{aligned} \tag{10}$$

where

$$\widetilde{h}_e = \frac{\widetilde{\varepsilon}\widetilde{\mu}'}{\widetilde{\chi}^2 - \widetilde{\varepsilon}\widetilde{\mu}} + \frac{1}{\rho}, \text{ and } \widetilde{h}_m = \frac{\widetilde{\varepsilon}'\widetilde{\mu}}{\widetilde{\chi}^2 - \widetilde{\varepsilon}\widetilde{\mu}} + \frac{1}{\rho},$$

$$\widetilde{g}_e = \omega^2(\widetilde{\chi}^2 + \widetilde{\varepsilon}\widetilde{\mu}) - \frac{1}{\rho^2} + \frac{1}{\rho}\frac{\widetilde{\varepsilon}\widetilde{\mu}'}{\widetilde{\chi}^2 - \widetilde{\varepsilon}\widetilde{\mu}},$$

$$\widetilde{g}_m = \omega^2(\widetilde{\chi}^2 + \widetilde{\varepsilon}\widetilde{\mu}) - \frac{1}{\rho^2} + \frac{1}{\rho}\frac{\widetilde{\varepsilon}'\widetilde{\mu}}{\widetilde{\chi}^2 - \widetilde{\varepsilon}\widetilde{\mu}},$$

$$\widetilde{f}_e = \frac{\widetilde{\chi}\widetilde{\varepsilon}'}{\rho\left(\widetilde{\chi}^2 - \widetilde{\varepsilon}\widetilde{\mu}\right)} \text{ and } \widetilde{f}_m = \frac{\widetilde{\chi}\widetilde{\mu}'}{\rho\left(\widetilde{\chi}^2 - \widetilde{\varepsilon}\widetilde{\mu}\right)},$$

$$\widetilde{k}_e = -2\omega^2\widetilde{\chi}\widetilde{\varepsilon} \text{ and } \widetilde{k}_m = -2\omega^2\widetilde{\chi}\widetilde{\mu},$$

satisfying the boundary and transmission conditions

$$\begin{gathered} u_e\big|_{r_0} = 0,\, u_m'\big|_{r_0} = 0,\, [u_e]\big|_r = 0,\, [u_m]\big|_r = 0, \\ \left[\frac{\widetilde{\chi}(\rho u_e)' - \widetilde{\mu}(\rho u_m)'}{\widetilde{\chi}^2 - \widetilde{\varepsilon}\widetilde{\mu}}\right]\Bigg|_r = 0, \quad \left[\frac{\widetilde{\chi}(\rho u_m)' - \widetilde{\varepsilon}(\rho u_e)'}{\widetilde{\chi}^2 - \widetilde{\varepsilon}\widetilde{\mu}}\right]\Bigg|_r = 0, \end{gathered} \tag{11}$$

the condition of boundedness of the field in any finite domain, and the condition of decay at infinity.

For $\rho > r$, we have $\widetilde{\varepsilon} = \varepsilon_0$, $\widetilde{\mu} = \mu_0$ and $\widetilde{\chi} = 0$; then from (10) we obtain the system

$$\begin{aligned} (\rho u_e')' - \rho\left(\kappa^2 + \frac{1}{\rho^2}\right) u_e &= 0, \\ (\rho u_m')' - \rho\left(\kappa^2 + \frac{1}{\rho^2}\right) u_m &= 0, \end{aligned}$$

where $\kappa^2 = \gamma^2 - \omega^2\varepsilon_0\mu_0$.

In view of the condition at infinity, a solution of this system is sought in the form

$$\begin{cases} u_e(\rho;\gamma,m) = C_1 K_1(\kappa\rho), \\ u_m(\rho;\gamma,m) = C_2 K_1(\kappa\rho), \end{cases} \tag{12}$$

where K_1 is the modified Bessel function (the Macdonald function) [21] and C_1 and C_2 are constants.

Remark 1. Due to the condition at infinity, we choose the following branch of the square root

$$\begin{aligned} \kappa = \sqrt{\gamma^2 - \varepsilon_c\mu_0} = \\ = \frac{1}{\sqrt{2}}\Bigg(\sqrt{|\gamma^2 - \varepsilon_0\mu_0| + \operatorname{Re}(\gamma^2 - \varepsilon_0\mu_0)} + \\ + i\operatorname{sign}\operatorname{Im}\left(\gamma^2 - \varepsilon_0\mu_0\right)\sqrt{|\gamma^2 - \varepsilon_0\mu_0| - \operatorname{Re}(\gamma^2 - \varepsilon_0\mu_0)}\Bigg). \end{aligned} \tag{13}$$

For $r_0 \le \rho \le r$, we have $\widetilde{\varepsilon} = \varepsilon(\rho)$, $\widetilde{\mu} = \mu(\rho)$ and $\widetilde{\chi} = \chi$, and from system (10) we obtain the system of differential equations

$$\begin{aligned} \mathscr{L}u_e := u_e'' + h_e u_e' + \left(g_e - \gamma^2\right) u_e &= f_m(\rho u_m)' + k_m u_m, \\ \mathscr{L}u_m := u_m'' + h_m u_m' + \left(g_e - \gamma^2\right) u_m &= f_e(\rho u_e)' + k_e u_e, \end{aligned} \tag{14}$$

where

$$h_e = \frac{\varepsilon\mu'}{\chi^2 - \varepsilon\mu} + \frac{1}{\rho}, \text{ and } h_m = \frac{\varepsilon'\mu}{\chi^2 - \varepsilon\mu} + \frac{1}{\rho},$$

$$g_e = \omega^2(\chi^2 + \varepsilon\mu) - \frac{1}{\rho^2} + \frac{1}{\rho}\frac{\varepsilon\mu'}{\chi^2 - \varepsilon\mu},$$

$$g_m = \omega^2(\chi^2 + \varepsilon\mu) - \frac{1}{\rho^2} + \frac{1}{\rho}\frac{\varepsilon'\mu}{\chi^2 - \varepsilon\mu},$$

$$f_e = \frac{\chi\varepsilon'}{\rho\left(\chi^2 - \varepsilon\mu\right)} \text{ and } f_m = \frac{\chi\mu'}{\rho\left(\chi^2 - \varepsilon\mu\right)},$$

$$k_e = -2\omega^2\chi\varepsilon \text{ and } k_m = -2\omega^2\chi\mu,$$

Knowing the solution in free space, the problem (10) - (11) can be reduced to an eigenvalue problem on the segment $[r_0,\, r]$.

Introduce the following

Definition 1. If there exist nontrivial functions u_e and u_m corresponding to some $\gamma \in \mathbb{C}$ such that these functions are the solutions (12) for $\rho > r$, solve system (14) for $r_0 \le \rho \le r$, and satisfy conditions (11), then γ is called a characteristic number of the problem.

Definition 2. The pair u_e and u_m, $|u_e|^2 + |u_m|^2 \not\equiv 0$ is called an eigenvector of the problem corresponding to the characteristic number $\gamma \in \mathbb{C}$.

3 Sobolev spaces and variational relation

We will look for solutions u_e and u_m of problem P_m in the Sobolev spaces

$$H_0^1(r_0, r) = \left\{f : f \in H^1(r_0, r),\, f(r_0) = 0\right\} \text{ and } H^1(r_0, r),$$

with the inner product and the norm

$$(f,g)_1 = \int_{r_0}^{r} \left(f'\overline{g}' + f\overline{g}\right) d\rho,$$

$$\|f\|_1^2 = (f,f)_1 = \int_{r_0}^{r} \left(|f'|^2 + |f|^2\right) d\rho.$$

Remark 2. Here we use the notation for the Sobolev space $H_0^1(r_0, r)$, which does not coincide with the standard one: in our case $f(r_0) = 0$ but generally $f(r) \neq 0$.

Let us give a variational formulation of problem P_m. Multiply equations (14) by arbitrary test functions $v_e \in H_0^1(r_0,r)$ and $v_m \in H^1(r_0,r)$ (one can assume that these functions are continuously differentiable in (r_0,r)) and apply Green's formula to obtain

$$\int_{r_0}^{r} \bar{v}\mathscr{L}u d\rho = \int_{r_0}^{r} u''\bar{v}d\rho + \int_{r_0}^{r} hu'\bar{v}d\rho - \int_{r_0}^{r} \left(\gamma^2 - g\right) u\bar{v}d\rho = $$
$$= u'\bar{v}\big|_{r_0}^{r} - \int_{r_0}^{r} u'\bar{v}'d\rho + \int_{r_0}^{r} hu'\bar{v}d\rho - \int_{r_0}^{r} \left(\gamma^2 - g\right) u\bar{v}d\rho, \tag{15}$$

where $u = u_j,\ v = v_j,\ h = h_j,\ g = g_j,\ j = e$ or m.

We separately apply formula (15) to the first and second equations in system (14) on the interval $[r_0,\ r]$ and add the results to verify that the sum of left-hand sides of Eqs. (14) satisfies the relation

$$\int_{r_0}^{r} (\bar{v}_e\mathscr{L}u_e + \bar{v}_m\mathscr{L}u_m)d\rho = -\gamma^2\int_{r_0}^{r} (u_e\bar{v}_e + u_m\bar{v}_m)d\rho -$$
$$- \int_{r_0}^{r} (u_e'\bar{v}_e' + u_m'\bar{v}_m')d\rho + \int_{r_0}^{r} (h_e u_e'\bar{v}_e + h_m u_m'\bar{v}_m)d\rho +$$
$$+ \int_{r_0}^{r} (g_e u_e\bar{v}_e + g_m u_m\bar{v}_m)d\rho + (u_e'(r)\bar{v}_e(r) + u_m'(r)\bar{v}_m(r)). \tag{16}$$

On the other hand, for the right-hand sides of the equations in system (14), we have

$$\int_{r_0}^{r} (\bar{v}_e\mathscr{L}u_e + \bar{v}_m\mathscr{L}u_m)d\rho =$$
$$= \int_{r_0}^{r} (f_e(\rho u_e)'\bar{v}_m + f_m(\rho u_m)'\bar{v}_e)d\rho +$$
$$+ \int_{r_0}^{r} (k_e u_e\bar{v}_m + k_m u_m\bar{v}_e)d\rho. \tag{17}$$

Given the solutions (12), we express the values of the normal derivatives at $\rho = r$ from relations (11) as follows:

$$\begin{aligned} u_e'(r) &= \kappa\left(\frac{\mu}{\varepsilon_0}F(\gamma) - \frac{1}{r}\right)u_e(r) + \kappa\frac{\chi}{\varepsilon_0}F(\gamma)u_m(r), \\ u_m'(r) &= \kappa\left(\frac{\varepsilon}{\mu_0}F(\gamma) - \frac{1}{r}\right)u_m(r) + \kappa\frac{\chi}{\mu_0}F(\gamma)u_e(r), \end{aligned} \tag{18}$$

where

$$F(\gamma) = -\frac{K_0(\kappa r)}{K_1(\kappa r)}.$$

From (16), taking into account (17) and (18), we obtain the sought *variational relation*

$$\gamma^2\int_{r_0}^{r} (u_e\bar{v}_e + u_m\bar{v}_m)d\rho +$$
$$+ \int_{r_0}^{r} (u_e'\bar{v}_e' + u_m'\bar{v}_m' + u_e\bar{v}_e + u_m\bar{v}_m)d\rho -$$
$$- \int_{r_0}^{r} ((g_e+1)u_e\bar{v}_e + (g_m+1)u_m\bar{v}_m)d\rho -$$
$$- \int_{r_0}^{r} (h_e u_e'\bar{v}_e + h_m u_m'\bar{v}_m)d\rho +$$
$$+ \int_{r_0}^{r} (f_e(\rho u_e)'\bar{v}_m + f_m(\rho u_m)'\bar{v}_e)d\rho +$$
$$+ \int_{r_0}^{r} (k_e u_e\bar{v}_m + k_m u_m\bar{v}_e)d\rho +$$
$$+ \left(\kappa\left(\frac{\mu}{\varepsilon_0}F(\gamma) - \frac{1}{r}\right)u_e(r) + \kappa\frac{\chi}{\varepsilon_0}F(\gamma)u_m(r)\right)\bar{v}_e(r) +$$
$$+ \left(\kappa\left(\frac{\varepsilon}{\mu_0}F(\gamma) - \frac{1}{r}\right)u_m(r) + \kappa\frac{\chi}{\mu_0}F(\gamma)u_e(r)\right)\bar{v}_m(r) = 0. \tag{19}$$

We note that variational relation (19) has been obtained for smooth functions $\bar{v}_e$ and $\bar{v}_m$.

Let $H = H_0^1(r_0,\ r) \times H^1(r_0,\ r)$ be the Cartesian product of the Hilbert spaces with the inner product and the norm

$$(\mathbf{u},\mathbf{v}) = (u_1,\ v_1)_1 + (u_2,\ v_2)_1,$$
$$\|\mathbf{u}\|^2 = \|u_1\|_1^2 + \|u_2\|_1^2; \mathbf{u},\mathbf{v} \in H,$$
$$\mathbf{u} = (u_1,\ u_2)^T,\ \mathbf{v} = (v_1,\ v_2)^T,$$
$$u_1, v_1 \in H_0^1(r_0,\ r),\ u_2, v_2 \in H^1(r_0,\ r).$$

Then the integrals occurring in (19) can be viewed as sesquilinear forms over the field $\mathbb{C}$ defined on the space H and depending on the arguments

$$\mathbf{u} = (u_e,\ u_m)^T,\ \mathbf{v} = (\bar{v}_e,\ \bar{v}_m)^T.$$

These forms t define some bounded linear operators $\mathrm{T} : H \to H$ by the formula [22]

$$\mathrm{t}(\mathbf{u},\ \mathbf{v}) = (\mathrm{T}\mathbf{u},\ \mathbf{v}),\ \forall \mathbf{v} \in H, \tag{20}$$

provided that the forms themselves are bounded,

$$|\mathrm{t}(\mathbf{u},\ \mathbf{v})| \le C\|\mathbf{u}\|\|\mathbf{v}\|.$$

The linearity follows from the linearity of the form in the first argument and the continuity from the estimates

$$\|\mathrm{T}\mathbf{u}\|^2 = \mathrm{t}(\mathbf{u},\mathrm{T}\mathbf{u}) \le C\|\mathbf{u}\|\|\mathrm{T}\mathbf{u}\|.$$

Consider the following sesquilinear forms and the corresponding operators:

$$\mathrm{k}(\mathbf{u},\ \mathbf{v}) := \int_{r_0}^{r} (u_e\overline{v}_e + u_m\overline{v}_m)d\rho \ = \ (\mathrm{K}\mathbf{u},\mathbf{v}),\ \forall \mathbf{v} \in H,$$

$$\mathrm{k}_1(\mathbf{u},\ \mathbf{v}) :=$$
$$= \int_{r_0}^{r} ((g_e+1)u_e\overline{v}_e + (g_m+1)u_m\overline{v}_m)d\rho = (\mathrm{K}_1\mathbf{u},\mathbf{v}),\ \forall \mathbf{v}\in H,$$

$$\widetilde{\mathrm{k}}(\mathbf{u},\ \mathbf{v}) := \int_{r_0}^{r} (k_e u_e\overline{v}_m + k_m u_m\overline{v}_e)d\rho = (\widetilde{\mathrm{K}}\mathbf{u},\mathbf{v}),\ \forall \mathbf{v}\in H,$$

$$\mathrm{a}(\mathbf{u},\ \mathbf{v}) :=$$
$$= \int_{r_0}^{r} (u_e'\overline{v}_e' + u_m'\overline{v}_m' + u_e\overline{v}_e + u_m\overline{v}_m)d\rho = (\mathrm{I}\mathbf{u},\mathbf{v}),\ \forall \mathbf{v}\in H,$$

$$\mathrm{b}_1(\mathbf{u},\ \mathbf{v}) := \int_{r_0}^{r} (h_e u_e'\overline{v}_e + h_m u_m'\overline{v}_m)d\rho = (\mathrm{B}_1\mathbf{u},\mathbf{v}),\ \forall \mathbf{v}\in H,$$

$$\mathrm{b}_2(\mathbf{u},\ \mathbf{v}) :=$$
$$= \int_{r_0}^{r} (f_e(\rho u_e)'\overline{v}_m + f_m(\rho u_m)'\overline{v}_e)d\rho = (\mathrm{B}_2\mathbf{u},\mathbf{v}),\ \forall \mathbf{v}\in H,$$

$$\mathrm{s}(\mathbf{u},\ \mathbf{v}) =$$
$$= \left(\kappa\left(\frac{\mu}{\varepsilon_0}F(\gamma) - \frac{1}{r}\right)u_e(r) + \kappa\frac{\chi}{\varepsilon_0}F(\gamma)u_m(r)\right)\overline{v}_e(r) +$$
$$+ \left(\kappa\left(\frac{\varepsilon}{\mu_0}F(\gamma) - \frac{1}{r}\right)u_m(r) + \kappa\frac{\chi}{\mu_0}F(\gamma)u_e(r)\right)\overline{v}_m(r) =$$
$$= (\mathrm{S}(\gamma)\mathbf{u},\mathbf{v}),\ \forall \mathbf{v}\in H. \quad (21)$$

Now variational problem (19) can be written in the operator form

$$(\mathrm{N}(\gamma)\mathbf{u},\ \mathbf{v}) = 0,\ \forall \mathbf{u}\in H$$

or, equivalently,

$$\mathrm{N}(\gamma)\mathbf{u} := \left(\gamma^2\mathrm{K} + \mathrm{I} - \mathrm{K}_1 - \mathrm{B}_1 + \mathrm{B}_2 + \widetilde{\mathrm{K}} + \mathrm{S}(\gamma)\right)\mathbf{u} = 0. \quad (22)$$

Equation (22) is the operator form of variational relation (19). The characteristic numbers and eigenvectors of operator N by definition coincide with the eigenvalues and eigenvectors of the original problem.

4 Properties of the operator-valued function

The problem on surface electromagnetic waves in an inhomogeneous open waveguide has been reduced to the study of the spectral properties of operator-valued function *rmN*.

We present the following statements about the properties of operators entering $\mathrm{N}(\gamma)$ (see the proof in [24]):

Statement 1. *The bounded operators* K, K_1 *and* $\widetilde{\mathrm{K}} : H \to H$ *are compact and* $\mathrm{K} > 0$.

Statement 2. *The operators* B_1 *and* $\mathrm{B}_2 : H \to H$ *are compact.*

Statement 3. *The operator* $\mathrm{S} : H \to H$ *is compact.*

Statement 4. *There exists a* $\widetilde{\gamma} \in \mathbb{R}$ *such that the operator* $\mathrm{N}(\widetilde{\gamma})$ *is continuously invertible; i.e., the resolvent set* $\rho(\mathrm{N}) := \{\gamma : \exists\, \mathrm{N}^{-1}(\gamma) : H \to H\}$ *of the operator function* $\mathrm{N}(\widetilde{\gamma})$ *is nonempty,* $\rho(\mathrm{N}) \neq \varnothing$.

Proof. Let $\gamma_0 = \omega\sqrt{\varepsilon_0\mu_0}$, then

$$\kappa = \sqrt{\gamma^2 - \omega^2\varepsilon_0\mu_0} = 0. \quad (23)$$

Consider operator-valued function $\mathrm{N}(\gamma)$ on the set $[\gamma_0, \gamma_0 + t)$, for some $t > 0$. This function is continuous on the indicated set due to (21), (22) and (23) and the asymptotics of functions as $z \to +0, (z \in\ R)$

$$K_0(z) \sim -\ln z,\ K_1(z) \sim \frac{1}{z},\ F(\gamma_0) = 0.$$

Then if the operator $\mathrm{N}^{-1}(\gamma_0) : H \to H$, then there is a $0 < t_0 < t$, such that the operator $\mathrm{N}^{-1}(\gamma_0 + t_0)$ will be bounded and $\gamma_0 + t_0 \in \rho(N)$. Thus, since $\mathrm{N}(\gamma)$ is a Fredholm operator, it suffices to prove that the equation $\mathrm{N}(\gamma_0)\mathbf{u} = 0$ has only a trivial solution.

Varying the functions v_e and v_m in (19), we see that (18) holds. Then for $\gamma = \gamma_0$ (respectively, $\kappa = 0$) we have $u_e'(r) = u_m'(r) = 0$. Substituting these expressions into conditions (11), it is easy check that for $\chi^2 \neq (\varepsilon(r) - \varepsilon_0)(\mu(r) - \mu_0)$ it is necessary that $u_e(r) = u_m(r) = 0$. We obtain a Cauchy problem for a system of second-order differential equations (10) with homogeneous (zero) initial conditions. Whence, due to the smoothness of the coefficients, we find that $u_e(\rho) \equiv u_m(\rho) \equiv 0$. Note that the point $\gamma_0 + t_0$ lies in the domain where operator-valued function $\mathrm{N}(\gamma)$ is holomorphic. Thus we have proved that the resolvent set is not empty. □

Theorem 1. *Operator-valued function* $\mathrm{N}(\gamma) : H \to H$ *is bounded, holomorphic, and Fredholm in the domain* $\Lambda = \mathbb{C}\backslash\widetilde{\Lambda}$, $\widetilde{\Lambda} = \{\gamma :\ \mathrm{Im}\,\gamma^2 = 0,\ \gamma^2 \leq \omega^2\varepsilon_0\mu_0\}$.

Proof. In the domain Λ, $\mathrm{N}(\gamma) : H \to H$ is bounded and holomorphic. Operator-valued function $\widetilde{\mathrm{N}}(\gamma)$ is Fredholm as the sum of invertible I and compact $\mathrm{K}, \mathrm{K}_1, \mathrm{B}_1, \mathrm{B}_2, \widetilde{\mathrm{K}}$ and S operators. □

Theorem 2. *The spectrum of operator-valued function* $\mathrm{N}(\gamma): H \to H$ *is discrete in the domain* Λ*; i.e., this function has finitely many characteristic numbers of finite algebraic multiplicity on any compact set* $K_0 \subset \Lambda$*.*

Proof. The assertion of the theorem is a corollary of Theorem 1 and properties of a holomorphic operator-valued function [25]. □

5 Numerical results

Using the projection method [26, 27] we reduce variational equation (19) to a system of algebraic equations.

First, split an interval $[r_0,\ r]$ into n subintervals with the length

$$h = \frac{r_0 - r}{n}.$$

Define a set of n subintervals

$$\Phi_i = [r_0 + (i-1)h,\ r_0 + (i+1)h],\ i = 1, ..., n-1$$

and

$$\Phi_n = [r_0 + (n-1)h,\ r],$$

and set of $n+1$ subintervals

$$\Psi_1 = [r_0,\ r_0 + h],$$

$$\Psi_j = [r_0 + (i-2)h,\ r_0 + ih],\ j = 2, ..., n$$

and

$$\Psi_{n+1} = [r_0 + (n-1)h,\ r].$$

These subintervals are called *base finite elements*.

The basis functions ϕ_i defined on Φ_i are

$$\phi_i = \begin{cases} \dfrac{\rho - r_0 - (i-1)h}{h}, & \rho < r_0 + ih, \\ -\dfrac{\rho - r_0 - (i+1)h}{h}, & \rho > r_0 + ih, \end{cases} ,i = \overline{1, n-1} \tag{24}$$

and

$$\phi_n = \frac{\rho - r + h}{l}; \tag{25}$$

The basis functions ψ_i defined on Ψ_i are

$$\psi_1 = -\frac{\rho^2 - 2r_0\rho + r_0^2 - h^2}{h^2}, \tag{26}$$

$$\psi_2 = \begin{cases} \dfrac{\rho^2 - 2r_0\rho + r_0^2}{h^2}, & \rho < r_0 + h, \\ -\dfrac{\rho - r_0 - 2h}{h}, & \rho > r_0 + h, \end{cases} \tag{27}$$

$$\psi_j = \begin{cases} \dfrac{\rho - r_0 - (i-2)h}{h}, & \rho < r_0 + (i-1)h, \\ -\dfrac{\rho - r_0 - ih}{h}, & \rho > r_0 + (i-1)h, \end{cases} ,j = \overline{3, n} \tag{28}$$

and

$$\psi_{n+1} = \frac{\rho - r + h}{h}. \tag{29}$$

Such basis functions take into account the physical nature of the problem under consideration.

We look for an approximate solution with real coefficients α_i and β_j such that

$$u_e = \sum_{i=1}^{n} \alpha_i \phi_i, \quad u_m = \sum_{j=1}^{n+1} \beta_j \psi_j. \tag{30}$$

Substituting functions u_e and u_m with representations (30) into the variational equation (19), we obtain a system of linear equations with respect to α_i and β_j (for a fixed value of γ)

$$A(\gamma)x = 0, \tag{31}$$

where matrices $A(\gamma)$ and x have the form

$$A = \begin{pmatrix} A_{ee}^{1,1} & \cdots & A_{ee}^{1,n} & A_{em}^{1,1} & \cdots & A_{em}^{1,n+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{ee}^{n,1} & \cdots & A_{ee}^{n,n} & A_{em}^{n,1} & \cdots & A_{em}^{n,n+1} \\ & & & & & \\ A_{me}^{1,1} & \cdots & A_{me}^{1,n} & A_{mm}^{1,1} & \cdots & A_{mm}^{1,n+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{me}^{n+1,1} & \cdots & A_{me}^{n+1,n} & A_{mm}^{n+1,1} & \cdots & A_{mm}^{n+1,n+1} \end{pmatrix},$$

$$x = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \\ \\ \beta_1 \\ \vdots \\ \beta_{n+1} \end{pmatrix},$$

and

$$A_{ee}^{i,j} = \gamma^2 \int\limits_{\Phi_i} \phi_i \phi_j d\rho + + \int\limits_{\Phi_i} \phi_i' \phi_j' d\rho - \int\limits_{\Phi_i} g_e \phi_i \phi_j d\rho - \int\limits_{\Phi_i} h_e \phi_i' \phi_j d\rho + + \kappa \left(\frac{\mu(r)}{\varepsilon_0} F(\gamma) - \frac{1}{r} \right) \phi_i(r) \phi_j(r),\ i, j = \overline{1, n};$$

$$A_{em}^{i,j} = \int\limits_{\Phi_i} \left(f_e (\rho \phi_i)' + + k_e \phi_i \right) \psi_j d\rho + + \kappa \frac{\chi}{\mu_0} F(\gamma) \phi_i(r) \psi_j(r),\ i = \overline{1, n}, j = \overline{1, n+1},$$

$$A_{me}^{i,j} = \int\limits_{\Psi_i} \left(f_m (\rho \psi_i)' + k_m \psi_i \right) \phi_j d\rho + + \kappa \frac{\chi}{\varepsilon_0} F(\gamma) \psi_i(r) \phi_j(r),\ i = \overline{1, n+1}, j = \overline{1, n},$$

$$A_{mm}^{i,j} = \gamma^2 \int\limits_{\Psi_i} \psi_i \psi_j d\rho +$$

$$+ \int\limits_{\Psi_i} \psi_i' \psi_j' d\rho - \int\limits_{\Psi_i} g_m \psi_i \psi_j d\rho - \int\limits_{\Psi_i} h_m \psi_i' \psi_j d\rho +$$

$$+ \kappa \left(\frac{\varepsilon(r)}{\mu_0} F(\gamma) - \frac{1}{r} \right) \psi_i(r) \psi_j(r), \; i,j = \overline{1, n+1}.$$

As a model problem, consider the following configuration of parameters: $r_0 = 1$, $r = 3$, $\varepsilon = 4 + \frac{\rho}{r}$, $\mu = 1$, $\varepsilon_0 = \mu_0 = 1$. Dispersion curves (graph of the dependence of the normalized propagation constant γ/ω on the frequency ω) are shown in the following figures.

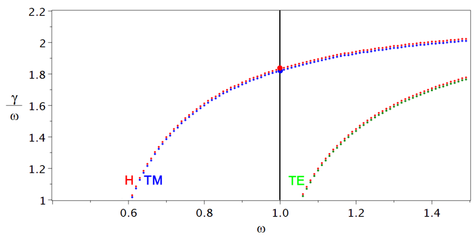

Figure 2. *Dispersion curves. The red dot lines correspond to the chiral filling of the waveguide $\chi = 0.0125$. Blue and green curves correspond to a waveguide filled with an inhomogeneous dielectric $\chi = 0$.*

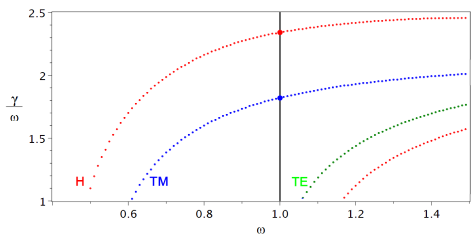

Figure 3. *Dispersion curves. The red dot lines correspond to the chiral filling of the waveguide $\chi = 1.25$. Blue and green curves correspond to a waveguide filled with an inhomogeneous dielectric $\chi = 0$.*

Figures 2 and 3 show that the spectrum of surface waves propagating in a waveguide filled with a chiral medium ($\chi = 0.00125$), at low frequencies coincide with the spectrum of a dielectric waveguide. However, with an increase of the chirality coefficient, the wave spectrum undergoes a noticeable deviation from the corresponding spectrum of an inhomogeneous dielectric waveguide.

Note that when the parameter χ is equal to zero, the problem splits into two independent problems of TE– and TM–polarized surface waves. The dispersion curves for such problem are shown in Figures 2 and 3 in blue and green color.

Next, we plot solutions to the problem inside the waveguide for the selected frequency $\omega = 1.0$. These solutions are presented in Figures 4 and 5.

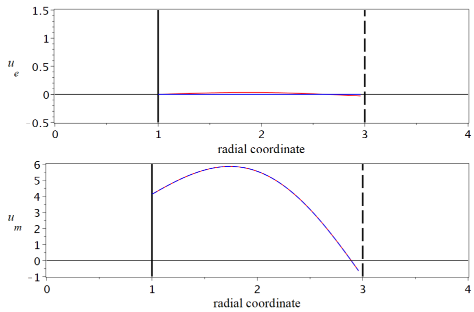

Figure 4. *Fields u_e and u_m. The red dots correspond to the chiral filling of the waveguide $\chi = 0.0125$. Blue lines corresponds to a waveguide filled with an inhomogeneous dielectric $\chi = 0$.*

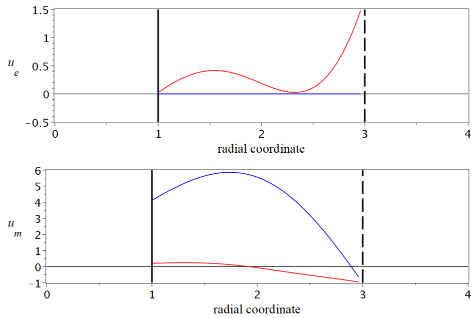

Figure 5. *Fields u_e and u_m. The red dots correspond to the chiral filling of the waveguide $\chi = 1.25$. Blue lines corresponds to a waveguide filled with an inhomogeneous dielectric $\chi = 0$.*

6 Conclusion

We have developed a method of analysis of the surface hybrid wave propagation in an inhomogeneous chiral metal-dielectric waveguide structure.

The definition of solution has been formulated using a variational relation. The variational problem has been reduced to the study of an operator-valued function. We have investigated all necessary properties of the operators entering the operator-valued function of the problem required for the analysis of its spectral properties. We have proved the discreteness of the spectrum of surface waves, which are characteristic numbers of the operator-valued function of the problem, and described the spectrum localization on the complex plane.

The performed study and its theoretical and calculation results ending up with a numerical method for determining surface waves have opened the way to creating a rigorous theory of the wave propagation in chiral waveguides.

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