



Complex waves in the inhomogeneous dielectric lossy Goubau line

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Abstract

The paper focuses on the propagation problem of the TE-waves in an open metal-dielectric inhomogeneous waveguide with losses. The propagation problem is reduced to a transmission eigenvalue problem for an ordinary differential equation. Spectral parameters of the problem are propagation constants of the waveguide. A numerical method for the determination of approximate eigenvalues is developed based on the shooting method. Using the method we can numerically find propagating, evanescent, and complex surface and leaky waves. Numerical results are presented.

1 Introduction

In this paper the electromagnetic TE-wave propagation in a conducting cylinder covered by a concentric dielectric layer, the Goubau line (GL), is investigated. The medium filling the waveguide is characterised by a isotropic inhomogeneous complex permittivity. In [1, 2, 3], the spectrum of symmetric surface modes in an open metal-dielectric waveguide structures was investigated.

Until now, the complex waves in GL have been widely studied [4, 5, 6, 7, 8, 9]. The results of [6, 7, 8] aimed at substantiation of the occurrence of complex waves in GL are largely based on a theoretical model [4] involving rigorous proof of the existence of complex waves. The approach set forth in [4] and detailed in [5] employs analysis of the function of several complex variables entering the dispersion equation for the complex wave spectra using Rouche's theorem. In [8] the existence of complex waves in GL is validated and the spectrum of surface complex waves is calculated as (regular) perturbations of the wave propagation constants with respect to the real parameter, imaginary part of the permittivity filling the GL covering dielectric layer. The technique employs numerical solution of the Cauchy problem obtained using the parameter-differentiation method. In [9] the existence of complex waves is demonstrated in a GL with a covering layer of metamaterial.

We reduce the electromagnetic problem under study to an eigenvalue problem where the spectral parameter is the propagation constant of TE-waves. A numerical method

for solving the problem is proposed which is based on the solution of an auxiliary Cauchy problem [10]. A similar approach allows one to numerically study propagation problems for other types of waves [11].

Correct classification of waves in GL is an important problem in the waveguide theory. The developed method allows one to classify electromagnetic TE-waves in GL and identify leaky and surface (depending on the condition at infinity) and propagating, evanescent, and complex waves (depending on the character of the propagation constant) [12, 13, 14]. Moreover, using the developed method it is possible to calculate waves in GL filled with metamaterial.

2 Statement of the problem

The waveguide $\Sigma := \{(\rho, \varphi) : r_0 \leq \rho \leq r, 0 \leq \varphi < 2\pi\}$ is placed in the cylindrical coordinate system $O\rho\varphi z$. Here r_0 and r are the radii of the internal and external cylinders, respectively. At the boundary $\rho = r_0$ there is a perfectly conducting screen. The concentric layer is filled with a isotropic inhomogeneous medium with losses. The external domain $\rho > r_0$ is filled with isotropic medium having constant permittivity $\epsilon_c \epsilon_0$, where $\epsilon_0 > 0$ is the permittivity of vacuum. We assume that $\mu = \mu_0$ everywhere, where $\mu_0 > 0$ is the permeability of vacuum.

We consider TE-polarized electromagnetic waves $(\mathbf{E}, \mathbf{H})e^{-i\omega t}$ propagating along GL Σ with a generating line parallel to the axis Oz , where $\omega > 0$ is a circular frequency; the complex amplitudes

$$\mathbf{E} = (0, E_\varphi(\rho)e^{i\gamma z}, 0), \quad \mathbf{H} = (H_\rho(\rho)e^{i\gamma z}, 0, H_z(\rho)e^{i\gamma z}),$$

and γ is the spectral parameter (propagation constant).

Maxwell's equations have the form

$$\nabla \times \mathbf{H} = -i\omega\epsilon_0\tilde{\epsilon}\mathbf{E}, \quad \nabla \times \mathbf{E} = i\omega\mu_0\mathbf{H}. \quad (1)$$

Thus complex amplitudes \mathbf{E} , \mathbf{H} satisfy equations (1), the boundary condition for the tangential electric field component on the perfectly conducting screen $E_\varphi|_{\rho=r_0} = 0$, and the transmission conditions for the tangential electric and magnetic field components on the discontinuity surface

of permittivity ($\rho = r$) $[E_\varphi]|_{\rho=r} = 0$, $[H_z]|_{\rho=r} = 0$, where $[f]|_{x_0} = \lim_{x \rightarrow x_0-0} f(x) - \lim_{x \rightarrow x_0+0} f(x)$.

Radiation conditions at infinity depend on the type of wave.

The permittivity in the whole space have the form

$$\tilde{\varepsilon} = \begin{cases} \varepsilon(\rho) + il, & r_0 \leq \rho \leq r, \\ \varepsilon_c, & \rho > r, \end{cases}$$

where $\varepsilon(\rho)$ is a continuous functions on segment $[r_0, r]$, i.e. $\varepsilon(\rho) \in C[r_0, r]$, l is a real constant.

The problem of determining waves is an eigenvalue problem for the Maxwell equations with spectral parameter γ .

The normal wave field in the waveguide can be represented using one scalar function $u := E_\varphi(\rho)$. Thus, the problem is reduced to finding tangential component u of the electric field. Everywhere below $(\cdot)' = \frac{d}{d\rho}$.

The following classification of waves is known [12, 13, 14].

Definition 1 *The propagating wave is characterised by real parameter γ .*

Definition 2 *The evanescent wave is characterised by pure imaginary parameter γ .*

Definition 3 *The complex wave is characterised by complex parameter γ such that $\Re\gamma\Im\gamma \neq 0$.*

Let us remark that propagation constant γ characterises the behaviour of a wave (propagating, evanescent, or complex) in the z -direction.

Definition 4 *The surface wave satisfies the condition*

$$u(\rho) \rightarrow 0, \quad \rho \rightarrow \infty. \quad (2)$$

Definition 5 *The leaky wave satisfies the condition*

$$u(\rho) \rightarrow \infty, \quad \rho \rightarrow \infty. \quad (3)$$

This yields that the classification of waves as surface or leaky depends on the behaviour of $u(\rho)$ at infinity.

Denote $k_0^2 := \omega^2 \mu_0 \varepsilon_0$. The propagation problem is reduced to the following eigenvalue problem for the tangential electric field component u : find $\gamma \in \mathbb{C}$ such that there exist non-trivial solutions of the differential equation

$$u'' + \rho^{-1}u' - \rho^{-2}u + (k_0^2\tilde{\varepsilon} - \gamma^2)u = 0, \quad (4)$$

satisfying the boundary conditions

$$u|_{\rho=r_0} = 0, \quad (5)$$

and the transmission conditions

$$[u]|_{\rho=r} = 0, \quad [u']|_{\rho=r} = 0. \quad (6)$$

For $\rho > r$, we obtain $\tilde{\varepsilon} = \varepsilon_c$; then from (4) we deduce

$$u'' + \rho^{-1}u' - \rho^{-2}u - \kappa^2u = 0. \quad (7)$$

In accordance with condition (2) or (3) (see definitions 4 and 5), we choose a solution of (7) for surface waves

$$u = C_1 K_1(\kappa\rho), \quad \rho > r, \quad (8)$$

and for leaky waves

$$u = C_2 I_1(\kappa\rho), \quad \rho > r, \quad (9)$$

respectively, where $\kappa = \sqrt{\gamma^2 - k_0^2 \varepsilon_c}$ and $\Re\kappa > 0$, C_1, C_2 are the constants, and K_m and I_m are the modified Bessel functions (Macdonald and Infeld functions) [15].

For $r_0 < \rho < r$, we have $\tilde{\varepsilon} = \varepsilon(\rho)$; thus from (4) we obtain

$$u'' + \rho^{-1}u' - \rho^{-2}u + (k_0^2(\varepsilon(\rho) + il) - \gamma^2)u = 0. \quad (10)$$

Definition 6 $\gamma \in \mathbb{C}$ is called propagation constant of the problem if there exists nontrivial solution u of (10) for $r_0 < \rho < r$, satisfying for $\rho > r$ (8) for surface waves and (9) for leaky waves, and conditions (5), (6).

3 Numerical method

Let us consider the Cauchy problem for equation (10) with the initial conditions

$$u(r_0) := 0, \quad u'(r_0) := A, \quad (11)$$

where A is a known real constant.

Now suppose that the Cauchy problem (10), (11) is globally and uniquely solvable on segment $[r_0, r]$ for given values r_0, r , and its solution continuously depends on parameter γ . Using the transmission condition on the boundary $\rho = r$ (6), one can obtain the dispersion equation for surface waves

$$\Delta_S(\gamma) \equiv K_1(\kappa r)u'(r) + (\kappa K_0(\kappa r) + r^{-1}K_1(\kappa r))u(r) = 0,$$

and the dispersion equation for leaky waves

$$\Delta_L(\gamma) \equiv I_1(\kappa r)u'(r) - (\kappa I_0(\kappa r) - r^{-1}I_1(\kappa r))u(r) = 0,$$

where quantities $u(r)$ and $u'(r)$ are obtained from the solution to the Cauchy problem (10), (11).

Let $\gamma = \alpha + i\beta$. Then we can write the dispersion equations for surface waves

$$\begin{cases} \Delta_1(\alpha, \beta) := \Re \Delta_S(\gamma) = 0, \\ \Delta_2(\alpha, \beta) := \Im \Delta_S(\gamma) = 0, \end{cases} \quad (12)$$

and for leaky waves

$$\begin{cases} \Delta_3(\alpha, \beta) := \Re \Delta_L(\gamma) = 0, \\ \Delta_4(\alpha, \beta) := \Im \Delta_L(\gamma) = 0, \end{cases} \quad (13)$$

as systems of real equations for determining parameters α and β .

Using the shooting method, we will solve numerically system of equations (12) to determine a pair (α, β) (see Fig. 1).

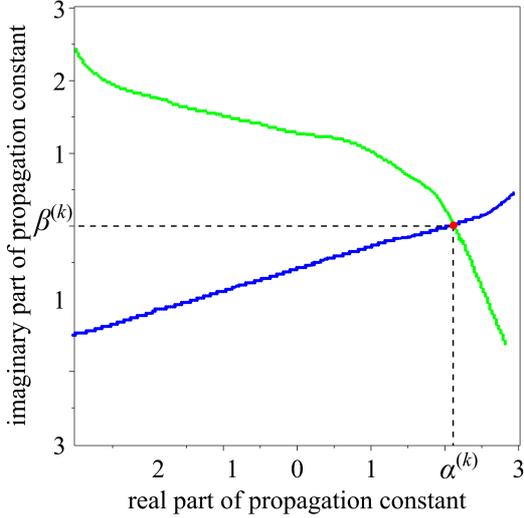


Figure 1. Dispersion curves: blue and green curves are solutions of, respectively, the first and the second equations of (12); red point is a solution of system (12).

Introduce a grid

$$\left\{ (\alpha^{(i)}, \beta^{(j)}) : \alpha^{(i)} = a_1 + i\tau_1, \beta^{(j)} = b_1 + j\tau_2, \right. \\ \left. i = \overline{0, n}, \tau_1 = \frac{a_2 - a_1}{n}, j = \overline{0, m}, \tau_2 = \frac{b_2 - b_1}{m} \right\}$$

with steps $\tau_1 > 0$, $\tau_2 > 0$, where a_1, a_2, b_1, b_2 are real fixed constants. Decreasing steps τ_1 and τ_2 , we can get arbitrarily accurate solutions.

Solving the Cauchy problem (10), (11) for each grid point, one obtains $u(r; \alpha^{(i)}, \beta^{(j)})$ and $u'(r; \alpha^{(i)}, \beta^{(j)})$, $i = \overline{0, n}$, $j = \overline{0, m}$. Note that solution $u(r; \alpha, \beta)$ is continuously dependent on parameters α and β . This means that if

$$\Delta_1(\alpha^{(i)}, \beta^{(j)}) \Delta_1(\alpha^{(i)}, \beta^{(j+1)}) \leq 0,$$

then there exists a point $(\alpha^{(i)}, \hat{\beta})$ in the plane $O\alpha\beta$, where $\hat{\beta} \in (\beta^{(j)}, \beta^{(j+1)})$, such that $\Delta_1(\alpha^{(i)}, \hat{\beta}) = 0$. Likewise, if

$$\Delta_1(\alpha^{(i)}, \beta^{(j)}) \Delta_1(\alpha^{(i+1)}, \beta^{(j)}) \leq 0,$$

then there exists a point $(\hat{\alpha}, \beta^{(j)})$ in the plane $O\alpha\beta$, where $\hat{\alpha} \in (\alpha^{(i)}, \alpha^{(i+1)})$, such that $\Delta_1(\hat{\alpha}, \beta^{(j)}) = 0$.

Continuing in the same manner, one finds a set of pairs $(\alpha^{(k)}, \beta^{(k)})$, where $k = \overline{0, p}$ and p is the number of the determined points. This set is presented as a curve in the plane $O\alpha\beta$ (the blue curve in Fig. 1).

Applying the same approach to the second equation of (12), one obtains another curve in the plane $O\alpha\beta$ (the green curve in Fig. 1). This curve is an approximate solution of the equation $\Delta_2(\alpha, \beta) = 0$. It is clear that the intersection point of the curves $(\hat{\alpha}, \hat{\beta})$ (the red point in Fig. 1) is an approximate solution of system (12). This implies that $\hat{\gamma} = \hat{\alpha} + i\hat{\beta}$ is a solution of the problem (a propagation constant of surface TE-wave). If $\hat{\alpha} = 0$ or $\hat{\beta} = 0$, we obtain an evanescent or propagating surface wave, respectively. In the case $\hat{\alpha}\hat{\beta} \neq 0$, propagation constant $\hat{\gamma}$ corresponds to a complex surface wave.

In the same way, we can obtain approximate solutions of system (13).

4 Numerical results

Figures 2 and 3 demonstrate the results of calculating the propagation constants for the problem of the TE-polarized wave propagation in GL filled with an inhomogeneous dielectric with losses. Propagating, evanescent surface TE-waves and propagating, evanescent, complex leaky TE-waves are determined numerically using the algorithm described in section 3.

The following values of parameters are used for calculations: $\varepsilon_c = 1$, $\varepsilon(\rho) = 4 + \frac{\rho}{3}$, $l = 0.05$, $A = 1$ V/mm, $k_0 = 1$ mm, $r_0 = 2$ mm, $r = 5$ mm, $a_1 = -3$, $a_2 = 3$, $b_1 = -3$, $b_2 = 3$, $\tau_1 = \tau_2 = 0.03$.

In Fig. 2 the solution to the problem of surface TE-waves in waveguide Σ is presented. The blue and green curves are solutions of, respectively, the first and the second equations of (12); the red intersection points correspond to the propagating surface TE-waves; the purple intersection points correspond to evanescent surface TE-waves.

In Fig. 3 the solution to the problem of leaky TE-waves in GL Σ is presented. The character of the curves is the same; the red intersection points correspond to the propagating leaky, the purple to the evanescent leaky, and the black to the complex leaky TE-waves.

5 Conclusion

The numerical method developed in this study complements the available techniques for analytical determination and calculation of the complex mode spectra in GL. The proposed approach enables one in particular to determine several different types of waves in GL using the same numerical algorithm which is a result important in various applications. The developed method can be generalized to GLs with multilayer dielectric covers.

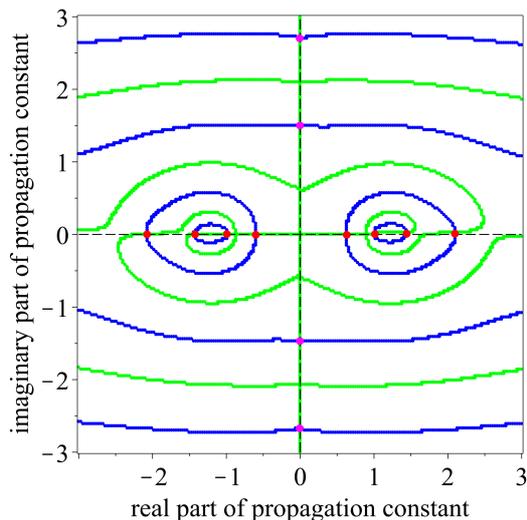


Figure 2. Dispersion curves for surface TE-waves.

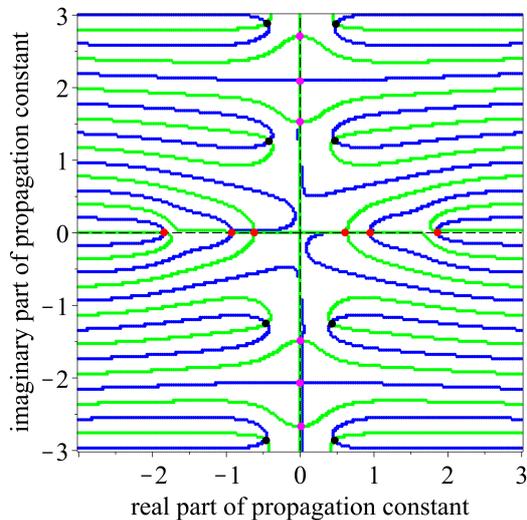


Figure 3. Dispersion curves for leaky TE-waves.

6 Acknowledgements

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References

[1] G. Goubau, “Surface waves and their application to transmission lines,” *Journal of Applied Physics*, **21**,

11, 1950, pp. 1119–1128, doi: 10.1063/1.1699553.

- [2] F. Harms, “Elektromagnetische wellen an einem draht mit isolierender zylindrischer hülle,” *Annalen der Physik*, **328**, 6, 1907, pp. 44–60, doi: 10.1002/andp.19073280603.
- [3] A. Sommerfeld, “Über die Fortpflanzung elektrodynamischer Wellen langs eines Drahtes,” *Annalen der Physik*, **303**, 2, 1899, pp. 233–290, doi: 10.1002/andp.18993030202.
- [4] Y. Shestopalov, “Complex waves in a dielectric waveguide,” *Wave Motion*, **82**, 2018, pp. 16-19.
- [5] Y. Shestopalov, E. Kuzmina, “On a Rigorous Proof of the Existence of Complex Waves in a Dielectric Waveguide of Circular Cross Section,” *Progress In Electromagnetics Research B*, **82**, 2018, pp. 137-164.
- [6] Y. Shestopalov, E. Kuzmina, “Waves in a Lossy Goubau Line,” *Proc. 2016 10th European Conference on Antennas and Propagation (EuCAP)*, 2016, pp. 1107-1111.
- [7] E.A. Kuzmina, Y.V. Shestopalov, “Complex Waves in Multi-Layered Metal-Dielectric Waveguides,” *Proc. 2018 International Conference on Electromagnetics in Advanced Applications (ICEAA)*, 2018, pp. 122-125.
- [8] E.A. Kuzmina, Y.V. Shestopalov, “Symmetric surface complex waves in Goubau Line,” *Cogent Engineering*, 2018, pp. 1-16.
- [9] E.A. Kuzmina, Y.V. Shestopalov, “Symmetric surface waves along a metamaterial dielectric waveguide and a perfectly conducting cylinder covered by a metamaterial layer,” *Advanced Electromagnetics*, **7**, 2, 2018.
- [10] V. Martynova, E. Smolkin, “Complex waves in the Goubau line,” *2021 34th General Assembly and Scientific Symposium of the International Union of Radio Science, URSI GASS 2021*, 2021, doi: 10.23919/URSIGASS51995.2021.9560566
- [11] V.Yu. Martynova, D.V. Valovik, “Electromagnetic Symmetric Hybrid Wave Propagation Problem in a Medium with Arbitrary Nonlinear Saturation,” *2019 Photonics and Electromagnetics Research Symposium - Fall*, 2019, pp. 2571–2578, doi: 10.1109/PIERS-Fall48861.2019.9021426.
- [12] L.A. Vainstein, “Electromagnetic Waves,” *Radio i svyaz*, Moscow, 1988.
- [13] R.E. Collin, “Field Theory of Guided Waves,” *Wiley-IEEE Press*, 1990.
- [14] L.B. Felsen, N. Marcuvitz, “Radiation and Scattering of Waves,” *Wiley-IEEE Press*, 1994.
- [15] M. Abramowitz and I. A. Stegun, “Handbook of Mathematical Functions,” *National Bureau of Standards*, Washington, D.C., 1972, 1046 p.