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The Predictive Posterior Probability Density Function for the Rectangular Probability Model and its Application to EMC and RF Measurements

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Abstract

The predictive posterior probability density function in the case of rectangular probability model is here derived. An example of application of this probability density function to a calculation of measurement uncertainty of a radiofrequency measurement is also illustrated. This probability density function is particularly useful when no prior information is available concerning the distribution of a quantity (e.g. electromagnetic field) over a region of space or time interval.

1. Introduction

The goal here is to provide justification to the use of the predictive posterior probability density function for describing the random variable associated with a quantity that unpredictably varies over a geometrical space or time interval. This is particularly useful for uncertainty calculations in the field of electromagnetic compatibility (EMC) measurements, where it is frequently required to generate a reference quantity (electric or magnetic field strength, power density, current intensity …) having a prescribed uniformity over a given region (length, surface or volume). The use of the predictive posterior probability density function is consistent with both the GUM [1] and GUM Supplement 1 [2] measurement uncertainty calculation procedures. Hence it can be immediately implemented by personnel operating in testing and calibration laboratories already familiar with the GUM and GUMS1.

Use is made here of Bayesian inference. Indeed, within this theoretical framework, appropriate statistical tools (namely, predictive posterior probability density functions) appear to be already available for the aim at hand. A new posterior probability density function is analytically derived that does not appear either in textbooks on Bayesian statistics or in the relevant scientific literature.

The paper is organized as follows. In section 2 the predictive posterior probability density function is recalled and its implementation for the rectangular probability models is provided. Application to assessment calculation of measurement uncertainty is described in section 3. Conclusion follows in section 4. An appendix provides details on mathematical derivation.

2. Predictive Posterior Probability Density Function

A sample of n observations of the random variable X is available and given by x_i , $i = 1, 2, ..., n$. One wants, on the basis of the known sample, to predict the probability distribution for a future observation $\tilde{x}$. It is assumed that the probability model is given but the parameters of the corresponding probability density function are unknown.

The case of the rectangular probability model is here considered. The posterior probability density function in the case of normal probability model is well known and available in textbooks (e.g. [3]).

Let us assume that X follows a rectangular probability density function with center c and half-width w . Let $x_a = \min\left[x_1, x_2, ..., x_n\right]$, $x_b = \max\left[x_1, x_2, ..., x_n\right]$ and $\delta = \left(x_b - x_a\right)/2$. No prior information is available about c and w. Then the joint density of c and w is [11]

$$f\left(c, w\right) = \frac{n\left(n-1\right)}{2\delta^2}\left(\frac{\delta}{w}\right)^{n+1}, \tag{1}$$

for $c - w < x_a$ and $c + w > x_b$, and $f\left(c, w\right) = 0$ otherwise. Let $u\left(z\right)$ be the step function defined as $u\left(z\right) = 0$ for $z < 0$ and $u\left(z\right) = 1$ for $z > 0$. Then the rectangular probability density function of center c and half-width w is given by

$$f\left(x; c, w\right) = \frac{1}{2w}\left[u\left(x - c + w\right) - u\left(x - c - w\right)\right]. \tag{2}$$

The predictive posterior probability density function of a future observation $\tilde{x}$ is $f\left(\tilde{x} \mid m, \delta\right)$ where $m = \left(x_a + x_b\right)/2$ and δ are the sufficient statistics. It is demonstrated in appendix A that

$$f\left(\tilde{x}\right) = \frac{n-1}{n+1}\frac{1}{2\delta} \cdot \begin{cases} \left[\dfrac{2}{1-\left(\tilde{x}-m\right)/\delta}\right]^n & \text{if } \tilde{x} < m-\delta \\ 1 & \text{if } m-\delta < \tilde{x} < m+\delta \\ \left[\dfrac{2}{1+\left(\tilde{x}-m\right)/\delta}\right]^n & \text{if } \tilde{x} > m+\delta \end{cases} \tag{3}$$

The shape of the predictive posterior probability density function for the rectangular probability model is shown in Fig. 1 in the case $n = 5$.

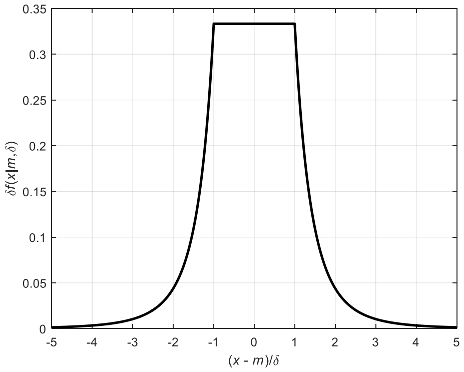

Figure 1. Predictive posterior probability density function for the rectangular probability model. Case *n* = 5.

The expected value of $\tilde{x}$ is (it is evident from the symmetry of (3) about *m*)

$$E\left\{\tilde{x}\right\} = m \tag{4}$$

and the variance of $\tilde{x}$ is (derivation is omitted for brevity)

$$\mathrm{Var}\left\{\tilde{x}\right\} = \frac{n-1}{n+1}\frac{\delta^2}{3} + \frac{2\delta^2}{n+1}\left[\frac{n^2-n+2}{(n-2)(n-3)}\right]. \tag{5}$$

In the limit $n \to \infty$ the predictive posterior tends to a rectangular probability density function having expected value m and variance $\delta^2/3$. Selected values of the ratio between $\mathrm{Var}\left\{\tilde{x}\right\}$ and the limit $\delta^2/3$ are reported in Table 1.

Table 1. Ratio between $\mathrm{Var}\left\{\tilde{x}\right\}$ and $\delta^2/3$ for selected values of *n*.

n	$\mathrm{Var}\left\{\tilde{x}\right\}/\left(\delta^2/3\right)$
4	3.40
5	1.89
6	1.48
7	1.30
8	1.21
9	1.15
10	1.12
15	1.04
20	1.02
50	1.00

The probability distribution function of $\tilde{x}$ is

$$F(\tilde{x}) = \frac{1}{n+1}\cdot\begin{cases}\left[\dfrac{2}{1-(\tilde{x}-m)/\delta}\right]^{n-1} & \text{if } \tilde{x} < m-\delta \\ \dfrac{1}{2}\left[n+1+(n-1)\cdot\dfrac{\tilde{x}-m}{\delta}\right] & \text{if } m-\delta < \tilde{x} < m+\delta \\ \left\{n+1-\left[\dfrac{2}{1+(\tilde{x}-m)/\delta}\right]^{n-1}\right\} & \text{if } \tilde{x} > m+\delta\end{cases} \tag{6}$$

The inverse of the probability distribution function, $\tilde{x} = F^{-1}(\rho)$, is readily obtained from (6) as

$$F^{-1}(\rho) = \begin{cases} m+\delta\left\{1-\dfrac{2}{\left[(n+1)\rho\right]^{\frac{1}{n-1}}}\right\} & \text{if } 0 < \rho < \dfrac{1}{n+1} \\ m+\delta\dfrac{n+1}{n-1}(2\rho-1) & \text{if } \dfrac{1}{n+1} < \rho < \dfrac{n}{n+1} \\ m+\delta\left\{\dfrac{2}{\left[(n+1)(1-\rho)\right]^{\frac{1}{n-1}}}-1\right\} & \text{if } \rho > \dfrac{n}{n+1}\end{cases} \tag{7}$$

For the purpose of random sampling from $F(\tilde{x})$ random values of ρ are drawn from a rectangular probability density function between 0 and 1 (see Annex C of [2]).

3. Measurement Uncertainty of the Power Density Transmitted by an Antenna in an Anechoic Chamber

An EM field is generated inside an anechoic chamber for the purpose of calibrating an electric field sensor at radiofrequency. The sensor is placed in front of a transmitting antenna fed by a signal generator. The power density of the EM field impinging on the sensor has to be determined. The measurement model equation linking the power density *S* (the measurand) with the net power *P* feeding the antenna, the antenna gain *G* and the distance *d* (along the boresight direction of the antenna) between the point where the sensor is placed and the reference point of the antenna, is as follows

$$S = P + G - 20\log(d) - 11.0 + X\,. \tag{8}$$

P and *G* in (8) are expressed in logarithmic units and *d* is the numerical value of the distance when expressed in meter. The quantities *P* and *d* are directly measured while *G* is obtained from a certificate of calibration of the antenna. The term *X* is a correction (in decibel) for the non-uniformity of the electric field strength over the volume where the sensor is placed. The random variables associated with *P*, *G*, *d* and *X* are independent. (8) can be rearranged as follows

$$S = S_0 + X\,, \tag{9}$$

where $S_0 = P + G - 20\log(d) - 11.0$.

Evidence supports the assignment of a normal probability density function to S_0, whose mean value is μ_0 and the variance is σ_0^2. A rectangular probability density function is instead assigned to the random variable *X*, whose upper and lower bounds are, a-priori, unknown. *X* is experimentally estimated by measuring the electric field strength at the $n = 8$ corners of a cubic volume. The deviation between the electric field strength measured at each of the eight points and μ_0 is calculated. Let the minimum deviation be x_a and the maximum deviation be x_b. Correspondingly, $m = (x_a + x_b)/2$ and $\delta = (x_b - x_a)/2$ are the parameters of the probability density function (3) which is assigned to the future observation of the correction $\tilde{x}$.

The following plausible values have been assumed in order to provide a numerical exemplification: $\mu_0 = 0.3$ dB(W/m^2) (corresponding to an electric field strength of approximately 20 V/m), $\sigma_0 = 0.4$ dB, $m = -0.2$ dB, $\delta = 1.0$ dB. The probability density functions for S_0, X and S are shown in Fig. 2 and in Fig. 3. The histogram plots have been generated by using the Monte Carlo method. Specifically note that (7) have been used in order to generate the histogram plot of X (and, as a consequence, of S, see (9)).

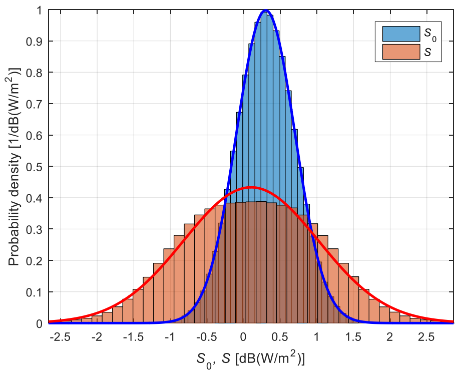

Figure 2. Histogram plots and normal probability density functions for S_0 and S (see the legend).

The exact probability density functions of S_0 (normal, blue continuous line) and X (see (3), blue continuous line) are plotted in Fig. 2 and Fig. 3, respectively. The normal probability density function having mean $\mu_0 + m$ and variance $\sigma_0^2 + s^2$ is also shown in Fig. 2 with red continuous line. The numerically computed 95 % coverage probability interval for S ranges from −1.6 dB(W/m^2) to 1.8 dB(W/m^2), while the corresponding interval obtained through the normal probability density function approximation (the red line in Fig. 2) ranges from −1.7 dB(W/m^2) to 1.9 dB(W/m^2).

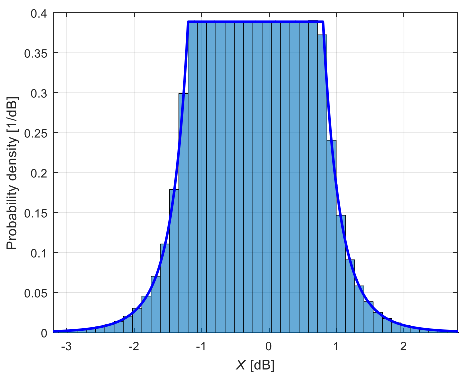

Figure 3. Histogram plot and probability density function (3) for X.

4. Conclusion

The use of the predictive posterior probability density function for the random variable associated with a quantity that randomly varies over a geometrical space or time interval is here supported through analysis and applications. The aim is ultimately to extend the application of the GUM and GUMS1 to measurands characterized by distribution of values, which is felt as an urgent need in conformity assessment and measurement uncertainty calculations in testing and calibration. The weakness of the proposed approach is the need to assign a defendable probability model to the measurand. The originally derived predictive posterior probability density function in the case of the rectangular probability model can be seen as a widely applicable "soft" parametric solution. Further investigation could be devoted to identify more flexible, adjustable probability models, than the normal and the rectangular ones, within which to derive the corresponding predictive posterior probability density function.

Appendix - Derivation of (3)

The posterior probability density function $f(\tilde{x})$ is, by definition [3], obtained as

$$f(\tilde{x}) = \iint_{D_{(c,w)}} f(\tilde{x}; c, w) \cdot f(c, w)\, dc\, dw \tag{10}$$

where $f(c,w)$ is given by (1) and $f(\tilde{x}; c, w)$ is given by (2). $D_{(c,w)}$ is the domain of the (c,w) values where $f(c,w)$ is different from zero.

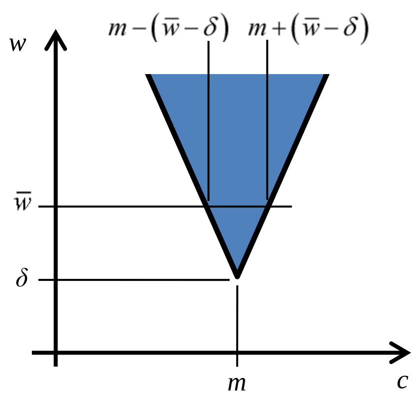

Figure A.1. Domain $D_{(c,w)}$ (shaded area).

Substituting (1) and (2) into (10) and taking into account that $D_{(c,w)}$ is the shaded area in Fig. A.1, we have

$$f(\tilde{x}) = \frac{n(n-1)}{2\delta^2} \cdot \int_{\delta}^{\infty} \left(\frac{\delta}{w}\right)^{n+1} \frac{1}{2w} \left\{ \int_{m-(w-\delta)}^{m+(w-\delta)} \left[u(\tilde{x}-c+w) - u(\tilde{x}-c-w) \right] dc \right\} dw \tag{11}$$

Consider the integral

$$I = \int_{m-(w-\delta)}^{m+(w-\delta)} \left[u\left(\tilde{x}-c+w\right) - u\left(\tilde{x}-c-w\right) \right] dc \,. \qquad (12)$$

We have

$$I = \begin{cases} \tilde{x}-(m+\delta)+2w & \text{if } \tilde{x} < m-\delta \text{ and } w > \dfrac{m+\delta-\tilde{x}}{2} \\ 2(w-\delta) & \text{if } m-\delta < \tilde{x} < m+\delta \text{ and } w > \delta \\ -\tilde{x}+(m-\delta)+2w & \text{if } \tilde{x} > m+\delta \text{ and } w > \dfrac{\tilde{x}-(m-\delta)}{2} \\ 0 & \text{otherwise} \end{cases} . \qquad (13)$$

The result (13) can be derived with the aid of Fig. A.2, showing the integrand function of c in (12), and considering that the following three possibilities for a non-zero value of I apply:

1. $m-(w-\delta) < \tilde{x}+w$ and $m+(w-\delta) > \tilde{x}+w$
2. $m+(w-\delta) < \tilde{x}+w$ and $m-(w-\delta) > \tilde{x}-w$
3. $m+(w-\delta) > \tilde{x}-w$ and $m-(w-\delta) < \tilde{x}-w$

Note that in any case the condition $w > \delta$ must be met (see Fig. 1).

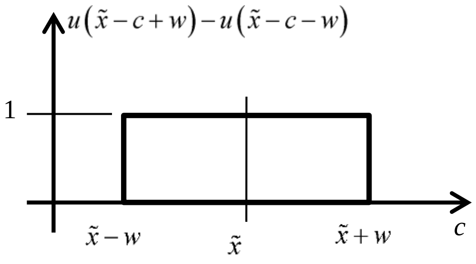

Figure A.2. Plot of the integrand function of integral I in (12).

Hence, if $\tilde{x} < m-\delta$ and $w > \dfrac{m+\delta-\tilde{x}}{2}$ then substituting the first row of (13) into (11) we have

$$\begin{aligned} f(\tilde{x}) &= \frac{n(n-1)}{2\delta^2} \int_{\frac{m+\delta-\tilde{x}}{2}}^{\infty} \left(\frac{\delta}{w}\right)^{n+1} \frac{\tilde{x}-(m+\delta)+2w}{2w}\, dw \\ &= \frac{n-1}{n+1}\frac{1}{2\delta}\left(\frac{2\delta}{m+\delta-\tilde{x}}\right)^{n} \end{aligned} . \qquad (14)$$

If $m-\delta < \tilde{x} < m+\delta$ and $w > \delta$ then substituting the second row of (13) into (11) we have

$$\begin{aligned} f(\tilde{x}) &= \frac{n(n-1)}{2\delta^2} \int_{\delta}^{\infty} \left(\frac{\delta}{w}\right)^{n+1} \frac{2w-2\delta}{2w}\, dw \\ &= \frac{n-1}{n+1}\frac{1}{2\delta} \end{aligned} . \qquad (15)$$

Finally, if $\tilde{x} > m+\delta$ and $w > \dfrac{\tilde{x}-(m-\delta)}{2}$ then substituting the third row of (13) into (11) we have

$$\begin{aligned} f(\tilde{x}) &= \frac{n(n-1)}{2\delta^2} \int_{\frac{\tilde{x}-(m-\delta)}{2}}^{\infty} \left(\frac{\delta}{w}\right)^{n+1} \frac{-\tilde{x}+(m-\delta)+2w}{2w}\, dw \\ &= \frac{n-1}{n+1}\frac{1}{2\delta}\left(\frac{2\delta}{\tilde{x}-(m-\delta)}\right)^{n} \end{aligned} . \qquad (16)$$

(3) immediately follows from (14), (15) and (16).

5. References

1. BIPM, IEC, IFCC, ILAC, ISO, IUPAC, IUPAP and OIML, 2008 Guide to the Expression of Uncertainty in Measurement, JCGM 100:2008, GUM 1995 with minor corrections.

2. BIPM, IEC, IFCC, ILAC, ISO, IUPAC, IUPAP and OIML, 2008 Supplement 1 to the 'Guide to the Expression of Uncertainty in Measurement'—Propagation of distributions using a Monte Carlo method JCGM 101:2008.

3. Andrew Gelman, John B. Carlin, Hal S. Stern, David B. Dunson, Aki Vehtari and Donald B. Rubin, Bayesian Data Analysis, Third edition, CRC Press, Boca Raton (FL), 2013.