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# Direct time domain analytical solution for the transient impedance of the horizontal grounding electrode

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## Abstract

The paper deals with a derivation of simple and efficient analytical solution for the calculation of transient impedance of the horizontal grounding electrode. The Generalized Telegrapher's Equation for spatial distribution of the scattered voltage in the frequency domain is used to derive the transient voltage at the feed point of the grounding electrode. Solution is obtained by using the Inverse Laplace Transform and Cauchy residue theorem. Transient impedance is determined as a ratio of time domain voltage and (lightning) current at the feeding point. Some illustrative numerical results are given.

## 1. Introduction

Transient impedance represents one of the most important parameters in a lightning protection system (LPS) design and its calculation is of paramount importance [1, 2]. Accuracy of the transient impedance calculation depends on the actual approximations used in developing the model and a solution method, respectively.

Different approaches have been used for the calculation of transient impedance of the grounding systems. One of the most commonly used methods is the Transmission Line Model (TLM) [3, 4]. However, the most accurate approach is the full wave electromagnetic model, based on related integral equations arising from the antenna theory and the thin-wire approximation [1, 5]. This approach ensures the accuracy of the results, but with a relatively high computational cost. The most important parameter to obtain is the current distribution induced along the electrode. Provided the electrode current is known, other parameters of interest (scattered voltage, transient impedance, radiated field) can be readily calculated [6]. For simpler (canonical) geometries, analytical solution of integro-differential equation can be obtained as discussed in [4, 7].

In this paper, a derivation of the direct time domain formula for the transient impedance of the horizontal grounding electrode is carried out. Transient impedance of a single grounding electrode is calculated using the frequency domain analytical solution for scattered voltage induced along the electrode. The obtained frequency domain expression is transformed into time domain via Inverse Laplace Transform and Cauchy residue theorem [7].

In the second part of the paper, a derivation of the direct time domain solution for transient impedance is outlined. Third part of the paper presents some numerical examples with the extensive analysis of the obtained results. Finally, the concluding remarks are given.

## 2. Theoretical background

### 2.1 Antenna model of the grounding electrode

Horizontal, perfectly conducting (PEC) grounding electrode of length $L$ and radius $a$, embedded in a lossy medium at depth $d$ and excited by an equivalent current source is considered [5], as shown in Figure 1.

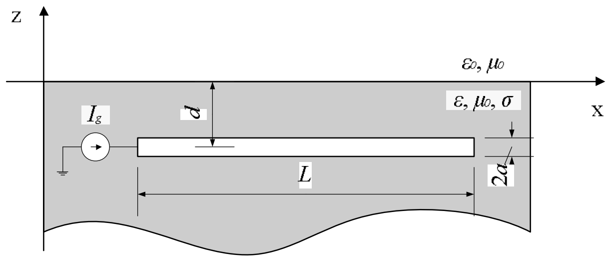


**Figure 1.** Horizontal grounding electrode.

The medium is characterized with electric permittivity $\varepsilon$ and conductivity $\sigma$. Dimensions of the electrode are assumed to satisfy the thin-wire approximation [8].

The current induced along the electrode $I(x')$ is governed by the homogeneous Pocklington equation which frequency domain variant is given with [8]

$$j\omega\frac{\mu_0}{4\pi}\int_0^L I(x')g(x,x')dx' - \frac{1}{j4\pi\omega\varepsilon_{eff}}\frac{\partial}{\partial x}\int_0^L\frac{\partial I(x')}{\partial x'}g(x,x')dx' = 0 , \quad (1)$$

where $\varepsilon_{eff}$ stands for the complex permittivity of the medium [9].

Green's function $g(x,x')$ can be expressed as [5]

$$g(x,x') = \frac{e^{-\gamma R_1}}{R_1} - \Gamma_{ref}\frac{e^{-\gamma R_2}}{R_2} . \quad (2)$$

Propagation constant of the medium is defined as $\gamma=\sqrt{j\omega\mu\sigma-\omega^2\mu\varepsilon}$ and distances $R_1$ and $R_2$ correspond to distances from the source and the image to the observation point, respectively. Presence of the earth-air interface is taken into account via the reflection coefficient arising from the Modified Image Theory [10]. Pocklington equation (1) can be solved analytically [5]

$$I(x)=I_g\frac{\sinh\left[\gamma(L-x)\right]}{\sinh(\gamma L)}. \tag{3}$$

The scattered voltage along the electrode is defined as an integral of the vertical component of the scattered electric field [9]. The Generalized Telegrapher's Equation for the spatial distribution of the scattered voltage is given as [9]

$$V^{sct}(x)=-\frac{1}{j4\pi\omega\varepsilon_{eff}}\int_0^L\frac{\partial I(x')}{\partial x'}g(x,x')dx'. \tag{4}$$

The expression for scattered voltage can be obtained by inserting (3) into (4), which yields

$$V^{sct}(x)=\frac{\gamma I_g}{j4\pi\omega\varepsilon_{eff}\sinh(\gamma L)}\int_0^L\cosh\left[\gamma(L-x')\right]g(x,x')dx'. \tag{5}$$

To calculate the transient impedance of the grounding electrode, the time domain relation for scattered voltage has to be determined by applying the Inverse Laplace Transform.

## 2.2 Time domain analytical solution procedure

To calculate the transient impedance, the knowledge of the voltage at the feeding point (i.e. $x$=0) of the electrode is required

$$V^{sct}(0)=\frac{\sqrt{s\mu\sigma+s^2\mu\varepsilon}}{4\pi(\varepsilon s+\sigma)\sinh\left(\sqrt{s\mu\sigma+s^2\mu\varepsilon}L\right)}\cdot\int_0^L\cosh\left[\gamma(L-x')\right]g(0,x')dx', \tag{6}$$

where the Green's function is given as

$$g(0,x')=\frac{e^{-\gamma\sqrt{x'^2+a^2}}}{\sqrt{x'^2+a^2}}+\frac{s\varepsilon_0(\varepsilon_r-1)+\sigma}{s\varepsilon_0(\varepsilon_r+1)+\sigma}\frac{e^{-\gamma\sqrt{x'^2+4d^2}}}{\sqrt{x'^2+4d^2}}. \tag{7}$$

Inverse Laplace Transform of (6) is determined by applying the Cauchy residue theorem [11]

$$f(t)=\sum_{k=1}^{n}\mathrm{Res}(s_k), \tag{8}$$

where the residues are determined according to

$$\mathrm{Res}(s_k)=\lim_{s\to s_k}(s-s_k)e^{ts}F(s). \tag{9}$$

The voltage function $V^{sct}(0)$ has infinite number of poles for $s=-\frac{\sigma}{\varepsilon}$ and $\sinh\left(\sqrt{s\mu\sigma+s^2\mu\varepsilon L}\right)=0$. Residuum for the first pole, according to (9), can be written as

$$\mathrm{Res}\left(-\frac{\sigma}{\varepsilon}\right)=\lim_{s\to-\frac{\sigma}{\varepsilon}}\frac{\left(s+\frac{\sigma}{\varepsilon}\right)\sqrt{s\mu\sigma+s^2\mu\varepsilon}e^{ts}}{4\pi(\varepsilon s+\sigma)\sinh\left(\sqrt{s\mu\sigma+s^2\mu\varepsilon}L\right)}\cdot\int_0^L\cosh\left[\gamma(L-x')\right]g(0,x')dx' \tag{10}$$

which can then be reduced to

$$\mathrm{Res}\left(-\frac{\sigma}{\varepsilon}\right)=\frac{1}{4\pi}e^{-\frac{\sigma}{\varepsilon}t}\int_0^L\left(\frac{1}{\sqrt{x'^2+a^2}}-\frac{1}{\sqrt{x'^2+4d^2}}\right)dx'\cdot\lim_{s\to-\frac{\sigma}{\varepsilon}}\frac{\left(s+\frac{\sigma}{\varepsilon}\right)\sqrt{s\mu\sigma+s^2\mu\varepsilon}}{(\varepsilon s+\sigma)\sinh\left(\sqrt{s\mu\sigma+s^2\mu\varepsilon}L\right)}. \tag{11}$$

Integral in (11) is readily calculated as

$$\int_0^L\left(\frac{1}{\sqrt{x'^2+a^2}}-\frac{1}{\sqrt{x'^2+4d^2}}\right)dx'=\ln\frac{2d\left(L+\sqrt{L^2+a^2}\right)}{a\left(L+\sqrt{L^2+4d^2}\right)}. \tag{12}$$

Assuming $L>>a$, residuum is determined as

$$\mathrm{Res}\left(-\frac{\sigma}{\varepsilon}\right)=\frac{1}{4\pi\varepsilon L}\ln\frac{4Ld}{a\left(L+\sqrt{L^2+4d^2}\right)}e^{-\frac{\sigma}{\varepsilon}t} \tag{13}$$

The condition for second (infinite number of) residues is

$$\sinh(\gamma L)=0. \tag{14}$$

Since $\sinh x=0$ for each $x=jn\pi$, $n=0,\pm1,\pm2,...$, the poles are determined as

$$s_{1,2n}=\frac{1}{2}\left(-b\pm\sqrt{b^2-4c_n}\right), \tag{15}$$

where $b$ and $c_n$ are defined with

$$b=\frac{\sigma}{\varepsilon},\qquad c_n=\frac{n^2\pi^2}{\mu\varepsilon L^2},\ n=1,2,3,... \tag{16}$$

Corresponding residues are calculated according to (9)

$$\mathrm{Res}(s_n)=\lim_{s\to s_n}(s-s_n)e^{ts}\frac{\sqrt{s\mu\sigma+s^2\mu\varepsilon}}{4\pi(\varepsilon s+\sigma)\sinh\left(\sqrt{s\mu\sigma+s^2\mu\varepsilon}L\right)}\int_0^L\cosh\left[\gamma(L-x')\right]g(0,x')dx' \tag{17}$$

Having performed some mathematical manipulations, (17) is given by

$$\mathrm{Res}(s_n) = \frac{(-1)^{n-1} n^2 \pi}{\pm \varepsilon \mu L^3 \sqrt{b^2 - 4c_n}\left(\sigma \pm \sqrt{b^2 - 4c_n}\right)} e^{ts_n} \int_0^L \cos \frac{n\pi (L - x')}{L} g(0, x', s_n)\, dx' \quad (18)$$

Combining (13) and (18), the following is obtained

$$v(0,t) = k_1 e^{-bt} + \sum_{n=1}^{\infty} k_2(n) e^{s_{1,2n}t} \int_0^L \cos \frac{n\pi (L - x')}{L} g(0, x', s_{1,2n})\, dx', \quad (19)$$

where

$$k_1 = \frac{1}{4\pi \varepsilon L} \ln \frac{4Ld}{a\left(L + \sqrt{L^2 + 4d^2}\right)}$$
$$k_2(n) = \frac{(-1)^{n-1} n^2 \pi}{\pm \varepsilon \mu L^3 \sqrt{b^2 - 4c_n}\left(\sigma \pm \sqrt{b^2 - 4c_n}\right)} \quad (20)$$
$$g(0, x', s_{1,2n}) = \frac{e^{-j\frac{n\pi}{L}\sqrt{x'^2 + a^2}}}{\sqrt{x'^2 + a^2}} + \frac{s_{1,2n}\varepsilon_0(\varepsilon_r - 1) + \sigma}{s_{1,2n}\varepsilon_0(\varepsilon_r + 1) + \sigma} \frac{e^{-j\frac{n\pi}{L}\sqrt{x'^2 + 4d^2}}}{\sqrt{x'^2 + 4d^2}}.$$

Equation (19) represents voltage for the impulse current excitation. In order to calculate the voltage and the impedance for a double exponential function, analytical convolution is performed, thus obtaining

$$v(0,t) = k_1 \left( \frac{e^{-bt} - e^{-\alpha t}}{\alpha - b} - \frac{e^{-bt} - e^{-\beta t}}{\beta - b} \right) + \sum_{n=1}^{\infty} k_2(n) \left( \frac{e^{s_{1,2n}t} - e^{-\alpha t}}{s_{1,2n} + \alpha} - \frac{e^{s_{1,2n}t} - e^{-\beta t}}{s_{1,2n} + \beta} \right) \cdot \int_0^L \cos \frac{n\pi (L - x')}{L} g(0, x', s_{1,2n})\, dx' \quad (21)$$

Transient impedance can be determined as [1]

$$z(t) = \frac{v(0,t)}{i(0,t)}. \quad (22)$$

Combining (21) and (22) the analytical expression for transient impedance is obtained

$$z(0,t) = \frac{1}{e^{-\alpha t} - e^{-\beta t}} \left[ k_1 \left( \frac{e^{-bt} - e^{-\alpha t}}{\alpha - b} - \frac{e^{-bt} - e^{-\beta t}}{\beta - b} \right) + \sum_{n=1}^{\infty} k_2(n) \left( \frac{e^{s_{1,2n}t} - e^{-\alpha t}}{s_{1,2n} + \alpha} - \frac{e^{s_{1,2n}t} - e^{-\beta t}}{s_{1,2n} + \beta} \right) \int_0^L \cos \frac{n\pi (L - x')}{L} g(0, x', s_{1,2n})\, dx' \right]. \quad (23)$$

## 3. Computational results

In this section, some illustrative numerical results for the transient impedance calculations are given. All the calculations are carried out for the grounding electrode with radius $a$=5 mm, buried at depth $d$=0.5 m in the ground with relative permittivity $\varepsilon_r$=10. Various values of electrode length, ground conductivity and lightning pulse are examined.

First set of calculations is carried out for the lightning current 1/10 μs pulse and ground conductivity $\sigma$=0.1 mS/m. The length of the electrode is varied as $L$=5, 10, 20 m. The results are shown in Figure 2.

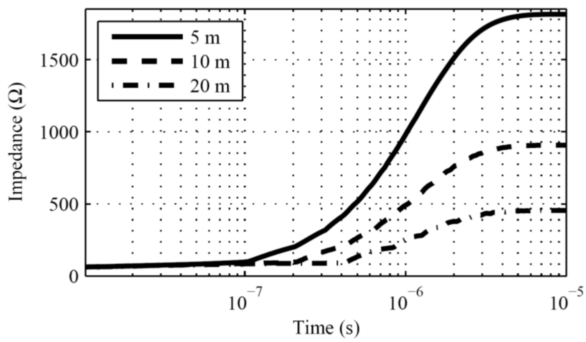


**Figure 2.** Transient impedance of the grounding electrode (1/10 μs pulse, $L$=5, 10, 20 m, $\sigma$=0.1 mS/m).

As can be seen in Figure 2, transient impedance is highly dependent on the length of the electrode. Even for a very low ground conductivity ($\sigma$=0.1 mS/m), increase of the electrode length can significantly reduce the steady state impedance. On the other hand, early time behavior of the impedance is not influenced by electrode length.

Figure 3 shows the results for the same configuration, but for higher ground conductivity ($\sigma$=1 mS/m). It can be seen that overall impedance is lower than in the previous example. It is worth noting that in the case of 20m long electrode, the steady state impedance is lower than the maximum impedance occurring at appx. $t$=50 ns. This property may have significant influence during the LPS design.

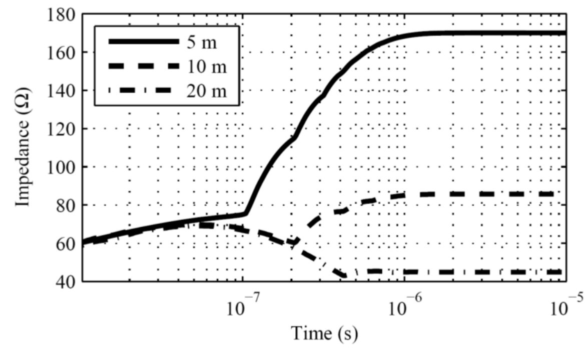


**Figure 3.** Transient impedance of the grounding electrode (1/10 μs pulse, $L$=5, 10, 20 m, $\sigma$=1 mS/m).

Following computational examples are calculated for 0.1/1 μs pulse, represents very fast pulse with high frequency content. Transient impedance for ground conductivity $\sigma$=0.5 mS/m is shown in Figure 4 and exhibits more oscillatory behavior which is a direct consequence of high frequency components of the input signal.

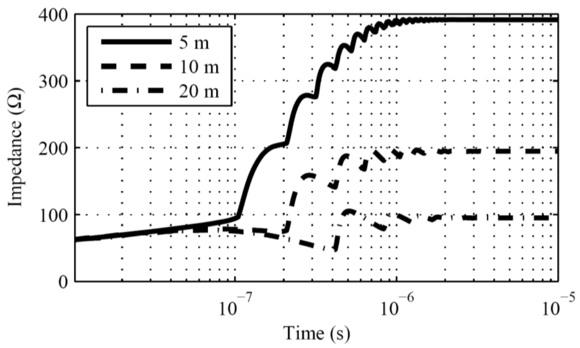


**Figure 4.** Transient impedance of the grounding electrode (0.1/1 μs pulse, $L$=5, 10, 20 m, $\sigma$=0.5 mS/m).

In Figure 5, same configuration is considered for the higher ground conductivity. Similar behavior of the transient impedance can be seen. When compared to the results for slower pulse (1/10 μs, shown in Figure 3) somewhat higher values of the impedance are observed. This can also be attributed to the high frequency content of the lightning current excitation.

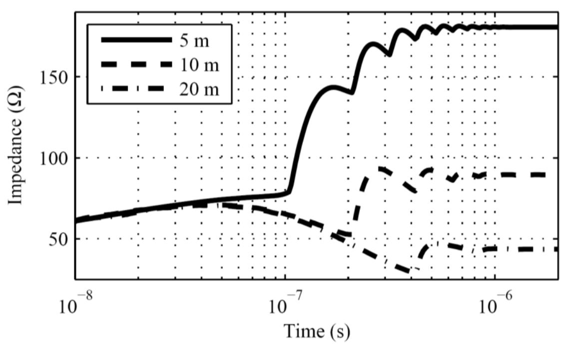


**Figure 5.** Transient impedance of the grounding electrode (0.1/1 μs pulse, $L$=5, 10, 20 m, $\sigma$=1 mS/m).

## 4. Concluding remarks

The paper proposes a simple and efficient approach for the calculation of transient impedance of the horizontal grounding electrode directly in the time. The approach is based on the antenna theory and related thin wire approximation. The formulation is based on the homogeneous Pocklington integro-differential equation and generalized Telegrapher's equation in the frequency domain. The direct time domain solution is obtained in the closed form. The principal advantage of the proposed analytical solution is to provide a rapid and satisfactory estimation of this complex phenomenon.

## 5. References


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